

ASYMMETRIC BLOW-UP FOR THE $SU(3)$ TODA SYSTEM

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ABSTRACT. We consider the so-called Toda system in a smooth planar domain under homogeneous Dirichlet boundary conditions. We prove the existence of a continuum of solutions for which both components blow-up at the same point. This blow-up behavior is asymmetric, and moreover one component includes also a certain global mass. The proof uses singular perturbation methods.

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1. INTRODUCTION

In this paper we consider the following version of the $SU(3)$ Toda system on a smooth bounded domain $\Omega \subset \mathbb{R}^2$:

$$\begin{cases} -\Delta u_1 = 2\rho_1 \frac{e^{u_1}}{\int_{\Omega} e^{u_1}} - \rho_2 \frac{e^{u_2}}{\int_{\Omega} e^{u_2}} & \text{in } \Omega, \\ -\Delta u_2 = 2\rho_2 \frac{e^{u_2}}{\int_{\Omega} e^{u_2}} - \rho_1 \frac{e^{u_1}}{\int_{\Omega} e^{u_1}} & \text{in } \Omega, \\ u_1 = u_2 = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.1)$$

Here ρ_1, ρ_2 are positive constants. This problem, and its counterpart posed on compact surfaces of \mathbb{R}^3 , has been very much studied in the literature. The Toda system has a close relationship with geometry, since it can be seen as the Frenet frame of holomorphic curves in \mathbb{CP}^N (see [14]). Moreover, it arises in the study of the non-abelian Chern-Simons theory in the self-dual case, when a scalar Higgs field is coupled to a gauge potential, see [12, 26, 27].

Problem (1.1) can also be seen as a natural generalization to systems of the classical mean field equation. With respect to the scalar case, the Toda system presents some analogies but also some different aspects, which have attracted the attention of a lot of mathematical research in recent years. Existence for the Toda system has been studied from a variational point of view in [4, 20, 16, 21], whereas blowing-up solutions have been considered in [2, 18, 19, 23, 24], for instance.

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The blow-up analysis for the solutions to (1.1) was performed in [15]; let us explain it in some detail. Assume that $u_n = (u_{1n}, u_{2n})$ is a blowing-up sequence of solutions of (1.1) with (ρ_{1n}, ρ_{2n}) bounded. Then, there exists a finite blow-up set $S = \{p_1, \dots, p_k\} \subset \Omega$ such that the solutions are bounded away from S . Concerning the points p_i , let us define the local masses:

$$\sigma_i = \lim_{r \rightarrow 0} \lim_{n \rightarrow +\infty} \rho_{in} \frac{\int_{B(p,r)} e^{u_{in}}}{\int_{\Omega} e^{u_{in}}}.$$

Then, the following scenarios are possible:

- a) Partial blow-up: $(\sigma_1, \sigma_2) = (4\pi, 0)$ or $(\sigma_1, \sigma_2) = (0, 4\pi)$. In such case, only one component is blowing up, and its profile is related to the entire solution of the Liouville problem in \mathbb{R}^2 .
- b) Asymmetric blow-up: $(\sigma_1, \sigma_2) = (4\pi, 8\pi)$ or $(\sigma_1, \sigma_2) = (8\pi, 4\pi)$. In this case, both components blow up and the local masses are different.
- c) Full blow-up: $(\sigma_1, \sigma_2) = (8\pi, 8\pi)$. In this case, both components blow up and the local masses are equal.

As a consequence of this study, the set of solutions is compact for any $\rho \in (\mathbb{R}^+)^2 \setminus \mathcal{C}$, where

$$\mathcal{C} = (4\pi\mathbb{N} \times \mathbb{R}^+) \cup (\mathbb{R}^+ \times 4\pi\mathbb{N}).$$

See [5, 15]. In other words, if blow-up occurs, at least one component u_{in} is quantized, and $\rho_{in} \rightarrow 4k\pi$ for some $k \in \mathbb{N}$.

Existence results of blowing-up solutions for the Toda system have been found in [2, 23, 18], which concern partial blow-up, asymmetric blow-up and full blow-up, respectively. In those papers, ρ_n converges to a single point of \mathcal{C} .

Our starting point is the following observation: *in the Toda system one expects the existence of continua of families of blowing-up solutions*. Indeed, if the Leray-Schauder degree of two adjacent squares of $\mathbb{R}^2 \setminus \mathcal{C}$ is different, then there must be blowing-up solutions for all points ρ in the common side.

In our preceding paper [8] we found continua of solutions exhibiting partial blow-up. The same type of solutions have been independently found in [17], where the authors use them to compute the degree for the Toda System when $\min\{\rho_1, \rho_2\} < 8\pi$.

In the present paper we prove the existence of continua of solutions which develop asymmetric blow-up. Indeed, given $\rho \in (4\pi, 8\pi)$, we are able to find solutions for values $(\rho_{1n}, \rho_{2n}) \rightarrow (8\pi, \rho)$ or, analogously, $(\rho_{1n}, \rho_{2n}) \rightarrow (\rho, 8\pi)$ (see Figure 1).

We will assume throughout the paper that Ω is k -symmetric for some $k > 2$ ($k \in \mathbb{N}$), i.e.

$$x \in \Omega \iff \mathfrak{R}_k(x) \in \Omega, \quad \text{where} \quad \mathfrak{R}_k(x) := \begin{pmatrix} \cos \frac{2\pi}{k} & \sin \frac{2\pi}{k} \\ -\sin \frac{2\pi}{k} & \cos \frac{2\pi}{k} \end{pmatrix} \cdot x, \quad k > 2. \quad (1.2)$$

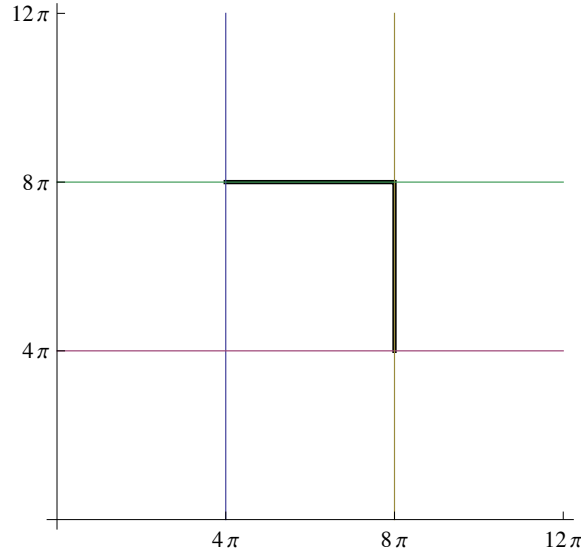


FIGURE 1. We find blowing-up solutions for which (ρ_1, ρ_2) converges to any point of the two marked segments (excluding their endpoints).

In this paper we prove the following theorem.

Theorem 1.1. *Let Ω be k -symmetric according to (1.2) and assume $0 \in \Omega$. Then, for any $\rho \in (4\pi, 8\pi)$, there exists a family of blowing-up solutions $(u_{1\lambda}, u_{2\lambda})$ of (1.1) for $\lambda \in (0, \lambda_0)$.*

Such a family has a unique blowing-up point at the origin as $\lambda \rightarrow 0$, and $(\sigma_1, \sigma_2) = (4\pi, 8\pi)$ (asymmetric blow-up). Moreover, the corresponding values $(\rho_{1\lambda}, \rho_{2\lambda})$ satisfy

$$\rho_{1\lambda} = \rho, \quad \rho_{2\lambda} \rightarrow 8\pi.$$

Concerning the asymptotic behavior of the solutions, if we make the change of variable

$$u_1 = 2v_1 - v_2, \quad u_2 = 2v_2 - v_1, \quad (1.3)$$

then the following holds:

$$(1) \quad v_{1\lambda}(x) = (-\log(\delta_1^2 + |x|^2) + 4\pi H(x, 0)) + \frac{1}{2}z(x) + o(1) \text{ in } H^1(\Omega)\text{-sense, where}^1$$

$$\delta_1 = \delta_1(\lambda) \sim \sqrt{\lambda} \text{ as } \lambda \rightarrow 0; \quad (1.4)$$

$$(2) \quad v_{2\lambda}(x) = -\log(\delta_2^4 + |x|^4) + 8\pi H(x, 0) + o(1) \text{ in } H^1(\Omega)\text{-sense, where}$$

$$\delta_2 = \delta_2(\lambda) \sim \sqrt[4]{\lambda} \text{ as } \lambda \rightarrow 0. \quad (1.5)$$

Here $H(x, y)$ denotes the regular part of the Green's function and z is the unique solution to the mean field equation

$$\begin{cases} \Delta z + 2(\rho - 4\pi) \frac{e^z}{\int_{\Omega} e^z} = 0 & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.6)$$

¹We use the notation \sim to denote quantities which in the limit $\lambda \rightarrow 0^+$ are of the same order.

Let us give a couple of comments on the assumptions of Theorem 1.1. For $\rho \in (4\pi, 8\pi)$, problem (1.6) admits a unique solution which is also nondegenerate, as has been proved in [3, 25]. Existence, uniqueness and nondegeneracy are the reasons for the restriction $\rho < 8\pi$. Moreover, the symmetry requirement in Theorem 1.1 is used to rule out the degeneracy of the radial solution of the singular Liouville problem, see Proposition 2.5 below.

Up to our knowledge, the only paper dealing with the existence of asymmetric blow-up for the Toda system is [23]. In the construction of [23], $\rho_n \rightarrow (4\pi, 8\pi)$, that is, there is no global mass. Our arguments follow some of the ideas of that paper, but some interesting differences have arose in our study. Observe that $v_{1\lambda}$ contains a peak around the origin which behaves as a solution of the regular Liouville problem, suitably rescaled. Moreover, $v_{2\lambda}$ is also blowing-up at the origin at a lower speed. In a certain sense, $e^{v_{1\lambda}}$ acts as a Dirac delta for $v_{2\lambda}$, and hence the limit profile of $v_{2\lambda}$ is the solution of a singular Liouville equation. Finally, $v_{1\lambda}$ contains also a macroscopic part, $z(x)$, which yields the global mass of the first component. This is one of the novelties with respect to [23]. At this scale the concentration effects of $v_{1\lambda}$ and $v_{2\lambda}$ cancel, and hence z takes the form of a solution of a regular Liouville problem posed in Ω .

A second difference is that our two scales of concentration (represented by the parameters δ_1 and δ_2) are *different from those in* [23]. This choice has been forced by the presence of the global mass, and implies that $\int_{\Omega} e^{u_1}$ remains bounded. This feature has another interesting implication; if we define $\tilde{u}_1 = u_1 + \log \frac{\rho_1}{\int_{\Omega} e^{u_1}}$, $\tilde{u}_2 = u_2 + \log \frac{\rho_2}{\int_{\Omega} e^{u_1}}$, we obtain solutions of the problem:

$$\begin{cases} -\Delta \tilde{u}_1 = 2e^{\tilde{u}_1} - e^{\tilde{u}_2} & x \in \Omega, \\ -\Delta \tilde{u}_2 = 2e^{\tilde{u}_2} - e^{\tilde{u}_1} & x \in \Omega, \\ \int_{\Omega} e^{\tilde{u}_i} < +\infty. \end{cases}$$

Those solutions are an example in which the singular set for both components reduces to the origin but only the second component diverges to $-\infty$ outside the origin. In other words, the generalization of the classical Brezis-Merle result [6] cannot involve both components in the Toda system.

The proofs use singular perturbation methods, which is based on the construction of suitable approximate solutions and on the study of the invertibility of the linearized operator. This study is a third difference with respect to [23]. Here, the first component has a dual behavior, global and local, which implies an interesting coupling between global and local terms, making the whole proof more involved.

The rest of the paper is organized as follows. Section 2 is devoted to some preliminary results, notation, and the definition of our approximating solution. Moreover, a more general version of Theorem 1.1 is stated there (see Theorem 2.2). The error up to which the approximating solution solves our problem is estimated in Section 3. In Section 4 we prove the solvability of the linearized problem. Finally, in Section 5, we prove the existence result by a contraction mapping argument, and we conclude the proof of Theorem 1.1 and Theorem 2.2.

2. PRELIMINARIES AND STATEMENT OF THE MAIN RESULT

In this section we will provide the *ansatz* for solutions of problem (1.1) and we will state our main result, which is a more general version of Theorem 1.1.

Motivated by the symmetry of the domain in assumption (1.2), we consider symmetric functions, i.e., functions satisfying

$$u = u \circ \mathfrak{R}_k, \quad \text{where } \mathfrak{R}_k \text{ is defined in (1.2).} \quad (2.1)$$

We define:

$$\mathcal{H}_k := \{u \in H_0^1(\Omega) : u \text{ satisfies (2.1)}\}.$$

In order to construct our solutions, we will use the solution z to the problem:

$$\begin{cases} \Delta z + 2(\rho - 4\pi) \frac{e^z}{\int_{\Omega} e^z} = 0 & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.2)$$

We shall need a nondegeneracy assumption on such solution, in the following form:

(H) Problem (2.2) is solvable in \mathcal{H}_k and the solution (if not unique, at least one of them) is nondegenerate. In other words, the linear problem

$$\begin{cases} \Delta \psi + 2(\rho - 4\pi) \frac{e^z \psi}{\int_{\Omega} e^z dx} - 2(\rho - 4\pi) \frac{e^z \int_{\Omega} e^z \psi dx}{\left(\int_{\Omega} e^z dx\right)^2} = 0 & \text{in } \Omega, \\ \psi = 0 & \text{on } \partial\Omega. \end{cases}$$

admits only the trivial solution in the space \mathcal{H}_k .

Remark 2.1. Problem (2.2) always admits a solution if $\rho < 8\pi$, which is easily found as a minimizer of its corresponding energy functional. Moreover, the solution is nondegenerate in this case, even without symmetry restrictions (see [25] for the case of a simply connected domain and [3] for the general case). If $\rho \geq 8\pi$ and Ω is the disk it is well-known that there is no solution of (2.2). For a non simply connected domain Ω , instead, problem (2.2) admits a solution for all $\rho \neq 4\pi(n+1)$, $n \in \mathbb{N}$, as shown in [7] (see also [10, 11] for a variational approach). In this case, though, one expects nondegeneracy results only for generic domains Ω .

In the rest of the paper we shall consider the following version of the Toda system, with fixed $\rho \in (4\pi, 8\pi)$ and sufficiently small $\lambda > 0$:

$$\begin{cases} \Delta u_1 + 2\rho \frac{e^{u_1}}{\int_{\Omega} e^{u_1}} - \lambda e^{u_2} = 0 & \text{in } \Omega, \\ \Delta u_2 + 2\lambda e^{u_2} - \rho \frac{e^{u_1}}{\int_{\Omega} e^{u_1}} = 0 & \text{in } \Omega, \\ u_1 = u_2 = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.3)$$

We now give a construction of a suitable approximate solution for (2.3). To this aim, for $\alpha \geq 2$, let us introduce the radially symmetric solutions of the singular Liouville problem

$$-\Delta w = |x|^{\alpha-2} e^w \quad \text{in } \mathbb{R}^2, \quad \int_{\mathbb{R}^2} |x|^{\alpha-2} e^{w(x)} dx < +\infty.$$

which are given by the one-parameter family of functions

$$w_\delta^\alpha(x) := \log 2\alpha^2 \frac{\delta^\alpha}{(\delta^\alpha + |x|^\alpha)^2} \quad x \in \mathbb{R}^2, \delta > 0.$$

The following quantization property holds:

$$\int_{\mathbb{R}^2} |x|^{\alpha-2} e^{w_\delta^\alpha(x)} dx = 4\pi\alpha. \quad (2.4)$$

To obtain a better first approximation, we need to modify the functions w_δ^α in order to satisfy the zero boundary condition. Precisely, we consider the projections Pw_δ^α onto the space $H_0^1(\Omega)$ of w_δ^α , where the projection $P : H^1(\mathbb{R}^N) \rightarrow H_0^1(\Omega)$ is defined as the unique solution of the problem

$$\Delta Pu = \Delta u \quad \text{in } \Omega, \quad Pu = 0 \quad \text{on } \partial\Omega.$$

We choose as initial approximation the following *ansatz*:

$$\begin{aligned} W_\lambda &= (W_{1\lambda}, W_{2\lambda}), \\ W_{1\lambda}(x) &:= Pw_1(x) - \frac{1}{2}Pw_2(x) + z(x), \\ W_{2\lambda}(x) &:= Pw_2(x) - \frac{1}{2}Pw_1(x) - \frac{1}{2}z(x), \end{aligned} \quad (2.5)$$

where

$$w_i(x) := w_{\delta_i}^{\alpha_i}(x) \quad \text{with } \alpha_1 := 2, \alpha_2 := 4, \quad (2.6)$$

and the values $\delta_i = \delta_i(\lambda)$ are defined as:

$$\delta_1 = \frac{1}{8} \sqrt{\frac{(\rho - 4\pi)\lambda}{\int_\Omega e^z dx}} e^{6\pi H(0,0) + \frac{z(0)}{4}}, \quad \delta_2 = \frac{1}{2} \sqrt[4]{\lambda} e^{3\pi H(0,0) - \frac{z(0)}{8}}. \quad (2.7)$$

Here $H(x, y)$ denotes the regular part of the Green's function of $-\Delta$ over Ω under homogeneous Dirichlet boundary conditions, namely

$$H(x, y) = G(x, y) - \frac{1}{2\pi} \log \frac{1}{|x - y|}.$$

By the maximum principle we easily deduce the following asymptotic expansion

$$\begin{aligned} Pw_i(x) &= w_i(x) - \log(2\alpha_i^2 \delta_i^{\alpha_i}) + 4\pi\alpha_i H(x, 0) + O(\delta_i^{\alpha_i}) \\ &= -2 \log(\delta_i^{\alpha_i} + |x|^{\alpha_i}) + 4\pi\alpha_i H(x, 0) + O(\delta_i^{\alpha_i}) \end{aligned} \quad (2.8)$$

uniformly for $x \in \Omega$.

We shall look for a solution to (2.3) in a small neighbourhood of the first approximation, namely a solution of the form

$$(u_{1\lambda}, u_{2\lambda}) = W_\lambda + \phi_\lambda,$$

where the rest term $\phi_\lambda := (\phi_{1\lambda}, \phi_{2\lambda})$ is small in $H^1(\Omega)$ -norm.

We are now in the position to state the main theorem of the paper.

Theorem 2.2. *Let Ω be k -symmetric according to (1.2) and $0 \in \Omega$. Assume that $\rho > 4\pi$ and condition (H) holds. Then, there exists $\lambda_0 > 0$ such that for any $\lambda \in (0, \lambda_0)$ there is $\phi_\lambda \in \mathcal{H}_k \times \mathcal{H}_k$ such that the couple $(W_{1\lambda} + \phi_{1\lambda}, W_{2\lambda} + \phi_{2\lambda})$ solves problem (2.3). Moreover, for any fixed $\varepsilon > 0$,*

$$\|\phi_\lambda\|_{(H_0^1(\Omega))^2} \leq \lambda^{\frac{1}{4}-\varepsilon} \quad \text{for } \lambda \text{ sufficiently small.}$$

As we shall see at the end of the paper, Theorem 1.1 follows quite directly from Theorem 2.2.

We end up this section by setting the notation and basic well-known facts which will be of use in the rest of the paper. We denote by $\|\cdot\|$ and $\|\cdot\|_p$ the norms in the space $H_0^1(\Omega)$ and $L^p(\Omega)$, respectively, namely

$$\|u\| := \|u\|_{H_0^1(\Omega)}, \quad \|u\|_p := \|u\|_{L^p(\Omega)} \quad \forall u \in H_0^1(\Omega). \quad (2.9)$$

Moreover, if $u = (u_1, u_2)$, we denote:

$$\|u\| = \|u_1\| + \|u_2\|, \quad \|u\|_p = \|u_1\|_p + \|u_2\|_p.$$

In next lemma we recall the well-known Moser-Trudinger inequality ([22, 28]).

Lemma 2.3. *There exists $C > 0$ such that for any bounded domain Ω in \mathbb{R}^2*

$$\int_{\Omega} e^{\frac{4\pi u^2}{\|u\|^2}} dx \leq C|\Omega| \quad \forall u \in H_0^1(\Omega),$$

where $|\Omega|$ stands for the measure of the domain Ω . In particular, for any $q \geq 1$

$$\|e^u\|_q \leq C^{\frac{1}{q}} |\Omega|^{\frac{1}{q}} e^{\frac{q}{16\pi} \|u\|^2} \quad \forall u \in H_0^1(\Omega).$$

For any $\alpha \geq 2$ we will make use of the Hilbert spaces

$$L_\alpha(\mathbb{R}^2) := \left\{ u \in W_{loc}^{1,2}(\mathbb{R}^2) : \left\| \frac{|y|^{\frac{\alpha-2}{2}}}{1+|y|^\alpha} u \right\|_{L^2(\mathbb{R}^2)} < +\infty \right\} \quad (2.10)$$

and

$$H_\alpha(\mathbb{R}^2) := \left\{ u \in W_{loc}^{1,2}(\mathbb{R}^2) : \|\nabla u\|_{L^2(\mathbb{R}^2)} + \left\| \frac{|y|^{\frac{\alpha-2}{2}}}{1+|y|^\alpha} u \right\|_{L^2(\mathbb{R}^2)} < +\infty \right\}, \quad (2.11)$$

endowed with the norms

$$\|u\|_{L_\alpha} := \left\| \frac{|y|^{\frac{\alpha-2}{2}}}{1+|y|^\alpha} u \right\|_{L^2(\mathbb{R}^2)} \quad \text{and} \quad \|u\|_{H_\alpha} := \left(\|\nabla u\|_{L^2(\mathbb{R}^2)}^2 + \left\| \frac{|y|^{\frac{\alpha-2}{2}}}{1+|y|^\alpha} u \right\|_{L^2(\mathbb{R}^2)}^2 \right)^{1/2}.$$

We denote by $\langle u, v \rangle_{L_\alpha}$ the natural scalar product in L_α .

Proposition 2.4. *The embedding $i_\alpha : H_\alpha(\mathbb{R}^2) \hookrightarrow L_\alpha(\mathbb{R}^2)$ is compact.*

Proof. See [13, Proposition 6.1]. □

As commented in the introduction, our proof uses the singular perturbation methods. For that, the nondegeneracy of the functions that we use to build our approximating solution is essential. Next proposition is devoted to the nondegeneracy of the entire solutions of the Liouville equation (regular and singular).

Proposition 2.5. *Assume that $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}$ satisfies (2.1) with $k > 2$ and solves the equation*

$$-\Delta\phi = 2\alpha^2 \frac{|y|^{\alpha-2}}{(1+|y|^\alpha)^2} \phi \text{ in } \mathbb{R}^2, \quad \int_{\mathbb{R}^2} |\nabla\phi(y)|^2 dy < +\infty, \quad (2.12)$$

with $\alpha = 2$ or $\alpha = 4$. Then there exists $\gamma \in \mathbb{R}$ such that

$$\phi(y) = \gamma \frac{1 - |y|^\alpha}{1 + |y|^\alpha}.$$

Proof. In [13, Theorem 6.1] it was proved that any solution ϕ of (2.12) is actually a bounded solution. Therefore we can apply the result in [9] to conclude that $\phi = c_0\phi_0 + c_1\phi_1 + c_2\phi_2$ for some $c_0, c_1, c_2 \in \mathbb{R}$, where

$$\phi_0(y) := \frac{1 - |y|^\alpha}{1 + |y|^\alpha}, \quad \phi_1(y) := \frac{|y|^{\frac{\alpha}{2}}}{1 + |y|^\alpha} \cos \frac{\alpha}{2}\theta, \quad \phi_2(y) := \frac{|y|^{\frac{\alpha}{2}}}{1 + |y|^\alpha} \sin \frac{\alpha}{2}\theta.$$

In the above definitions we have used polar coordinates. Note that ϕ_0 is radially symmetric and hence it satisfies (2.1); thus, $c_1\phi_1 + c_2\phi_2$ must satisfy (2.1). Observe now that

$$c_1\phi_1(y) + c_2\phi_2(y) = A \frac{|y|^{\frac{\alpha}{2}}}{1 + |y|^\alpha} \sin \left(\frac{\alpha}{2}\theta + \theta_0 \right)$$

with $A = \sqrt{c_1^2 + c_2^2}$, $\theta_0 \in \mathbb{R}$. Since $k > 2$, we get that $A = 0$ and, consequently, $c_1 = c_2 = 0$, concluding the proof. \square

Remark 2.6. *The validity of Proposition 2.5 is the main reason for the symmetry requirement (1.2).*

In our estimates throughout the paper, we will frequently denote by $C > 0$, $c > 0$ fixed constants, that may change from line to line, but are always independent of the variable under consideration. We also use the notations $O(1)$, $o(1)$, $O(\lambda)$, $o(\lambda)$ to describe the asymptotic behaviors of quantities in a standard way.

3. ESTIMATE OF THE ERROR TERM

The goal of this section is to provide an estimate of the error up to which the couple $(W_{1\lambda}, W_{2\lambda})$ solves system (2.3). First of all, we perform the following estimate.

Lemma 3.1. *Define*

$$E_{1\lambda} := 2\rho \frac{e^{W_{1\lambda}}}{\int_{\Omega} e^{W_{1\lambda}}} - e^{w_1} - 2(\rho - 4\pi) \frac{e^z}{\int_{\Omega} e^z}, \quad E_{2\lambda} := 2\lambda e^{W_{2\lambda}} - |x|^2 e^{w_2}.$$

For any $p \geq 1$ the following holds

$$\|E_{1\lambda}\|_p = O(\lambda^{\frac{2-p}{2p}}), \quad \|E_{2\lambda}\|_p = O(\lambda^{\frac{2-p}{4p}}).$$

Proof. By (2.8) we compute

$$e^{W_{1\lambda}} = e^{Pw_1 - \frac{1}{2}Pw_2 + z} = \frac{\delta_2^4 + |x|^4}{(\delta_1^2 + |x|^2)^2} e^{z + O(\delta_1^2) + O(\delta_2^4)} = \frac{\delta_2^4 + |x|^4}{(\delta_1^2 + |x|^2)^2} e^z (1 + O(\lambda))$$

and, since $\frac{|x|^4}{(\delta_1^2 + |x|^2)^2} = 1 + O(\frac{\delta_1^2}{\delta_1^2 + |x|^2})$, we deduce

$$e^{W_{1\lambda}} = \frac{\delta_2^4}{(\delta_1^2 + |x|^2)^2} e^z (1 + O(\lambda)) + e^z + O\left(\frac{\lambda}{\lambda + |x|^2}\right). \quad (3.1)$$

Then we scale the first term as $x = \delta_1 y$ and, owing to $e^{z(\delta_1 y)} = e^{z(0)}(1 + O(\delta_1 y))$, we get

$$e^{W_{1\lambda}(x)} = \frac{\delta_2^4}{\delta_1^4(1 + |y|^2)^2} e^{z(0)}(1 + O(\lambda) + O(\sqrt{\lambda}|y|)) + e^{z(x)} + O\left(\frac{\lambda}{\lambda + |x|^2}\right). \quad (3.2)$$

Therefore, using that $\int_{\frac{\Omega}{\delta_1}} \frac{1}{(1+|y|^2)^2} = \int_{\mathbb{R}^2} \frac{dy}{(1+|y|^2)^2} + O(\sqrt{\lambda}) = \pi + O(\sqrt{\lambda})$, we arrive at

$$\int_{\Omega} e^{W_{1\lambda}} dx = \frac{\delta_2^4}{\delta_1^2} e^{z(0)} \pi + \int_{\Omega} e^z dx + O(\sqrt{\lambda}). \quad (3.3)$$

In view of the choice of δ_1, δ_2 in (2.7), we have that (3.3) can be rewritten in the following two forms:

$$\int_{\Omega} e^{W_{1\lambda}} dx = \frac{\delta_2^4}{\delta_1^2} \frac{\rho}{4} e^{z(0)} + O(\sqrt{\lambda}) = \frac{\rho}{\rho - 4\pi} \int_{\Omega} e^z dx + O(\sqrt{\lambda}), \quad (3.4)$$

and, combining (3.4) with (3.2),

$$\begin{aligned} 2\rho \frac{e^{W_{1\lambda}(x)}}{\int_{\Omega} e^{W_{1\lambda}}} &= \frac{8(1 + O(\sqrt{\lambda}) + O(\sqrt{\lambda}|y|))}{\delta_1^2(1 + |y|^2)^2} + 2(\rho - 4\pi) \frac{e^{z(x)}}{\int_{\Omega} e^z} (1 + O(\sqrt{\lambda})) + O\left(\frac{\lambda}{\lambda + |x|^2}\right) \\ &= e^{w_1(x)} + O\left(\frac{1}{\sqrt{\lambda}(1 + |y|)^3}\right) + 2(\rho - 4\pi) \frac{e^{z(x)}}{\int_{\Omega} e^z} (1 + O(\sqrt{\lambda})) + O\left(\frac{\lambda}{\lambda + |x|^2}\right). \end{aligned}$$

Therefore, it follows that

$$\left\| 2\rho \frac{e^{W_{1\lambda}}}{\int_{\Omega} e^{W_{1\lambda}}} - e^{w_1} - 2(\rho - 4\pi) \frac{e^z}{\int_{\Omega} e^z} \right\|_p = O(\lambda^{\frac{2-p}{2p}}).$$

We turn our attention to the estimate of $e^{W_{2\lambda}}$; by (2.8) we obtain

$$e^{W_{2\lambda}} = e^{Pw_2 - \frac{1}{2}Pw_1 - \frac{1}{2}z} = \frac{\delta_1^2 + |x|^2}{(\delta_2^4 + |x|^4)^2} e^{12\pi H(x,0) - \frac{z}{2}} (1 + O(\lambda)). \quad (3.5)$$

Now we scale $x = \delta_2 y$:

$$\begin{aligned} e^{W_{2\lambda}(x)} &= \frac{\delta_1^2 + \delta_2^2|y|^2}{\delta_2^8(1 + |y|^4)^2} e^{12\pi H(\delta_2 y, 0) - \frac{z(\delta_2 y)}{2}} (1 + O(\lambda)) \\ &= \frac{|y|^2}{\delta_2^6(1 + |y|^4)^2} e^{12\pi H(\delta_2 y, 0) - \frac{z(\delta_2 y)}{2}} (1 + O(\lambda)) + O\left(\frac{1}{\lambda(1 + |y|^4)^2}\right) \\ &= \frac{|y|^2}{\delta_2^6(1 + |y|^4)^2} e^{12\pi H(0,0) - \frac{z(0)}{2}} (1 + O(\lambda) + O(\sqrt[4]{\lambda}|y|)) + O\left(\frac{1}{\lambda(1 + |y|^4)^2}\right). \end{aligned}$$

The choice of δ_2 in (2.7) yields

$$2\lambda e^{W_{2\lambda}(x)} = |x|^2 e^{w_2(x)} + O\left(\frac{1}{\sqrt[4]{\lambda}(1 + |y|^4)}\right)$$

and we conclude

$$\|2\lambda e^{W_{2\lambda}} - |x|^2 e^{w_2}\|_p = O\left(\lambda^{\frac{2-p}{4p}}\right).$$

□

Now we are going to estimate the error term R_λ ,

$$R_\lambda := (R_{1\lambda}, R_{2\lambda}), \quad (3.6)$$

where

$$\begin{aligned} R_{1\lambda} &:= -\Delta W_{1\lambda} - 2\rho \frac{e^{W_{1\lambda}}}{\int_\Omega e^{W_{1\lambda}}} + \lambda e^{W_{2\lambda}}, \\ R_{2\lambda} &:= -\Delta W_{2\lambda} - 2\lambda e^{W_{2\lambda}} + \rho \frac{e^{W_{1\lambda}}}{\int_\Omega e^{W_{1\lambda}}}. \end{aligned}$$

Lemma 3.2. *Let R_λ be as in (3.6). Then for any $p \in [1, 2]$ we have*

$$\|R_\lambda\|_p = O\left(\lambda^{\frac{1}{4}\frac{2-p}{p}}\right).$$

Proof. By (2.5), recalling that z solves (2.2), we have

$$\begin{aligned} R_{1\lambda} &= -\Delta \left(Pw_1 - \frac{1}{2}Pw_2 + z \right) - 2\rho \frac{e^{W_{1\lambda}}}{\int_\Omega e^{W_{1\lambda}}} + \lambda e^{W_{2\lambda}} \\ &= \left(e^{w_1} - 2\rho \frac{e^{W_{1\lambda}}}{\int_\Omega e^{W_{1\lambda}}} + 2(\rho - 4\pi) \frac{e^z}{\int_\Omega e^z} \right) - \frac{1}{2} (|x|^2 e^{w_2} - 2\lambda e^{W_{2\lambda}}). \end{aligned}$$

Analogously

$$R_{2\lambda} = (|x|^2 e^{w_2} - 2\lambda e^{W_{2\lambda}}) - \frac{1}{2} \left(e^{w_1} - 2\rho \frac{e^{W_{1\lambda}}}{\int_\Omega e^{W_{1\lambda}}} + 2(\rho - 4\pi) \frac{e^z}{\int_\Omega e^z} \right)$$

and the thesis follows by applying Lemma 3.1. □

4. ANALYSIS OF THE LINEARIZED OPERATOR

Let us consider the following linear problem: given $h_1, h_2 \in \mathcal{H}_k$, find functions ϕ_1, ϕ_2 satisfying

$$\begin{cases} -\Delta \phi_1 + \lambda e^{W_{2\lambda}} \phi_2 - 2\rho \left[\frac{e^{W_{1\lambda}} \phi_1}{\int_\Omega e^{W_{1\lambda}} dx} - \frac{e^{W_{1\lambda}} \int_\Omega e^{W_{1\lambda}} \phi_1 dx}{\left(\int_\Omega e^{W_{1\lambda}} dx \right)^2} \right] = \Delta h_1, \\ -\Delta \phi_2 - 2\lambda e^{W_{2\lambda}} \phi_2 + \rho \left[\frac{e^{W_{1\lambda}} \phi_1}{\int_\Omega e^{W_{1\lambda}} dx} - \frac{e^{W_{1\lambda}} \int_\Omega e^{W_{1\lambda}} \phi_1 dx}{\left(\int_\Omega e^{W_{1\lambda}} dx \right)^2} \right] = \Delta h_2, \\ \phi_1, \phi_2 \in \mathcal{H}_k. \end{cases} \quad (4.1)$$

Proposition 4.1. *For every $p \in (1, 2)$ there exist $\lambda_0 > 0$ and $C > 0$ such that for any $\lambda \in (0, \lambda_0)$, any $h_1, h_2 \in \mathcal{H}_k$ and any $\phi_1, \phi_2 \in \mathcal{H}_k$ solutions of (4.1), the following holds*

$$\|\phi_1\| + \|\phi_2\| \leq C |\log \lambda| (\|h_1\| + \|h_2\|).$$

Proof. We argue by contradiction. Assume that there exist $p \in (1, 2)$, sequences $\lambda_n \rightarrow 0$, $h_{i_n} \in \mathcal{H}_k$ and $\phi_{i_n} \in \mathcal{H}_k$ for $i = 1, 2$, which solve (4.1) and

$$\|\phi_{1_n}\| + \|\phi_{2_n}\| = 1, \quad (4.2)$$

$$|\log \lambda_n|(\|h_{1_n}\| + \|h_{2_n}\|) \rightarrow 0. \quad (4.3)$$

We define $\widetilde{\Omega}_{i_n} := \frac{\Omega}{\delta_{i_n}}$ and

$$\tilde{\phi}_{i_n}(y) := \begin{cases} \phi_{i_n}(\delta_{i_n} y) & \text{if } y \in \widetilde{\Omega}_{i_n} \\ 0 & \text{if } y \in \mathbb{R}^2 \setminus \widetilde{\Omega}_{i_n} \end{cases}.$$

In what follows at many steps of the arguments we will pass to a subsequence, without further notice. Moreover, for notational convenience, we avoid double subscripts and we will simply write W_1, W_2 in the place of $W_{1\lambda_n}, W_{2\lambda_n}$.

We split the remaining argument into five steps.

Step 1. We will show that

$$\begin{aligned} \tilde{\phi}_{1_n} & \text{ is bounded in } H_2(\mathbb{R}^2), \\ \tilde{\phi}_{2_n} & \text{ is bounded in } H_4(\mathbb{R}^2) \end{aligned}$$

(see (2.11)).

It is immediate to check that

$$\int_{\mathbb{R}^2} |\nabla \tilde{\phi}_{i_n}|^2 dy = \int_{\Omega} |\nabla \phi_{i_n}|^2 dx \leq 1, \quad i = 1, 2. \quad (4.4)$$

Next, we multiply the first equation in (4.1) by ϕ_{2_n} , the second equation by $2\phi_{2_n}$; then we integrate over Ω and sum up to obtain

$$\begin{aligned} 3\lambda_n \int_{\Omega} e^{W_2} \phi_{2_n}^2 dx &= 2 \int_{\Omega} |\nabla \phi_{2_n}|^2 dx + \int_{\Omega} \nabla \phi_{1_n} \nabla \phi_{2_n} dx + \int_{\Omega} \nabla h_{1_n} \nabla \phi_{2_n} dx \\ &+ 2 \int_{\Omega} \nabla h_{2_n} \nabla \phi_{2_n} dx \end{aligned}$$

which implies, by (4.2)–(4.3),

$$\lambda_n \int_{\Omega} e^{W_2} \phi_{2_n}^2 dx \leq C. \quad (4.5)$$

So, Lemma 3.1 gives $\int_{\Omega} |x|^2 e^{w_2} \phi_{2_n}^2 \leq C$ or, equivalently,

$$\int_{\mathbb{R}^2} \frac{|y|^2}{(1 + |y|^4)^2} \tilde{\phi}_{2_n}^2 dy \leq C.$$

Combining this with (4.4), we deduce that $\tilde{\phi}_{2_n}$ is bounded in the space $H_4(\mathbb{R}^2)$.

We now consider the first component. First, let $\psi_1 \in C_c^\infty(\Omega \setminus \{0\})$, $\psi_1 \geq 0$ and ψ_1 not identically zero. Then we multiply the first equation in (4.1) by ψ_1 , we integrate over Ω

and we get

$$\begin{aligned} & \int_{\Omega} \nabla \phi_{1n} \nabla \psi_1 dx - 2\rho \left[\frac{\int_{\Omega} e^{W_1} \phi_{1n} \psi_1 dx}{\int_{\Omega} e^{W_1} dx} - \frac{\int_{\Omega} e^{W_1} \psi_1 dx \int_{\Omega} e^{W_1} \phi_{1n} dx}{\left(\int_{\Omega} e^{W_1} dx\right)^2} \right] \\ & + \lambda_n \int_{\Omega} e^{W_2} \phi_{2n} \psi_1 dx = - \int_{\Omega} \nabla h_{1n} \nabla \psi_1 dx. \end{aligned} \quad (4.6)$$

We observe that by (3.1) and (3.5)

$$e^{W_1} \rightarrow e^z, \quad e^{W_2} \rightarrow \frac{1}{|x|^2} e^{12\pi H(x,0) - \frac{z}{2}} \text{ uniformly on compact sets of } \Omega \setminus \{0\} \quad (4.7)$$

and therefore, recalling (3.4), (4.6) yields

$$\int_{\Omega} e^{W_1} \phi_{1n} dx = O(1). \quad (4.8)$$

Now we multiply the first equation in (4.1) by $2\phi_{1n}$, the second equation by ϕ_{1n} , we integrate over Ω and sum up to obtain

$$3\rho \left[\frac{\int_{\Omega} e^{W_1} \phi_{1n}^2 dx}{\int_{\Omega} e^{W_1} dx} - \frac{\left(\int_{\Omega} e^{W_1} \phi_{1n} dx\right)^2}{\left(\int_{\Omega} e^{W_1} dx\right)^2} \right] = O(1). \quad (4.9)$$

By taking into account (4.8), we get $\int_{\Omega} e^{W_1} \phi_{1n}^2 dx = O(1)$ which implies, by Lemma 3.1, $\int_{\Omega} e^{w_1} \phi_{1n}^2 = O(1)$ or, equivalently,

$$\int_{\mathbb{R}^2} \frac{\tilde{\phi}_{1n}^2}{(1 + |y|^2)^2} dy = O(1)$$

and the thesis follows.

Step 2. We will show that, for some $\gamma_1, \gamma_2 \in \mathbb{R}$,

$$\tilde{\phi}_{1n} \rightarrow \gamma_1 \frac{1 - |y|^2}{1 + |y|^2} \text{ weakly in } H_2(\mathbb{R}^2) \text{ and strongly in } L_2(\mathbb{R}^2), \quad (4.10)$$

$$\tilde{\phi}_{2n} \rightarrow \gamma_2 \frac{1 - |y|^4}{1 + |y|^4} \text{ weakly in } H_4(\mathbb{R}^2) \text{ and strongly in } L_4(\mathbb{R}^2), \quad (4.11)$$

and

$$\phi_{1n} \rightarrow 0 \text{ weakly in } H_0^1(\Omega) \text{ and strongly in } L^q(\Omega) \text{ for any } q \geq 2. \quad (4.12)$$

Step 1 and Proposition 2.4 give

$$\tilde{\phi}_{1n} \rightarrow f \text{ weakly in } H_2(\mathbb{R}^2) \text{ and strongly in } L_2(\mathbb{R}^2).$$

Moreover, since $\|\phi_{1n}\| \leq 1$,

$$\phi_{1n} \rightarrow g \text{ weakly in } H_0^1(\Omega) \text{ and strongly in } L^q(\Omega) \text{ for any } q \geq 2.$$

Observe that each $\tilde{\phi}_{1n}$ satisfies (2.1) owing to the definition of \mathcal{H}_k , then f also satisfies (2.1). Let $\tilde{\psi}_1 \in C_c^\infty(\mathbb{R}^2)$ and set $\psi_{1n} = \tilde{\psi}_1(\frac{x}{\delta_{1n}}) \in C_c^\infty(\Omega)$, for large n . We multiply the first equation in (4.1) by ψ_{1n} , we integrate over Ω and we get

$$\begin{aligned} & \int_{\widetilde{\Omega}_{1n}} \nabla \tilde{\phi}_{1n} \nabla \tilde{\psi}_1 dy - 2\rho \left[\frac{\int_{\Omega} e^{W_1} \phi_{1n} \psi_{1n} dx}{\int_{\Omega} e^{W_1} dx} - \frac{\int_{\Omega} e^{W_1} \psi_{1n} dx \int_{\Omega} e^{W_1} \phi_{1n} dx}{\left(\int_{\Omega} e^{W_1} dx\right)^2} \right] \\ & + \lambda_n \int_{\Omega} e^{W_2} \phi_{2n} \psi_{1n} dx = - \int_{\Omega} \nabla h_{1n} \nabla \psi_{1n} dx. \end{aligned} \quad (4.13)$$

According to Lemma 3.1 we have

$$\begin{aligned} \frac{2\rho}{\int_{\Omega} e^{W_1}} \int_{\Omega} e^{W_1} \phi_{1n} dx &= \int_{\Omega} e^{w_1} \phi_{1n} dx + 2 \frac{\rho - 4\pi}{\int_{\Omega} e^z} \int_{\Omega} e^z \phi_{1n} + o(1) \\ &= 8 \int_{\mathbb{R}^2} \frac{\tilde{\phi}_{1n}}{(1 + |y|^2)^2} dy + 2 \frac{\rho - 4\pi}{\int_{\Omega} e^z} \int_{\Omega} e^z \phi_{1n} dx + o(1) \\ &= 8 \int_{\mathbb{R}^2} \frac{f}{(1 + |y|^2)^2} dy + 2 \frac{\rho - 4\pi}{\int_{\Omega} e^z} \int_{\Omega} e^z g dx + o(1). \end{aligned} \quad (4.14)$$

Similarly,

$$\frac{2\rho}{\int_{\Omega} e^{W_1}} \int_{\Omega} e^{W_1} \phi_{1n} \psi_{1n} dx = 8 \int_{\mathbb{R}^2} \frac{f \tilde{\psi}_1}{(1 + |y|^2)^2} dy + o(1)$$

and

$$\frac{2\rho}{\int_{\Omega} e^{W_1}} \int_{\Omega} e^{W_1} \psi_{1n} dx = 8 \int_{\mathbb{R}^2} \frac{\tilde{\psi}_1}{(1 + |y|^2)^2} dy + o(1)$$

Observe that $|x|^2 e^{w_2} \leq C$ in the support of ψ_{1n} ; then, again by Lemma 3.1 we get $\lambda_n e^{W_2} \psi_{1n} \rightarrow 0$ in $L^q(\Omega)$ for all $q \in (1, 2)$, and so can estimate:

$$\lambda_n \int_{\Omega} e^{W_2} \phi_{2n} \psi_{1n} dx = o(1).$$

Finally, by (4.3), using that $\int_{\Omega} |\nabla \psi_{1n}|^2 = \int_{\mathbb{R}^2} |\nabla \tilde{\psi}_1|^2$,

$$\int_{\Omega} |\nabla h_{1n} \nabla \psi_{1n}| dx = O(\|h_{1n}\|) = o(1). \quad (4.15)$$

Therefore, we may pass to the limit in (4.13) to obtain

$$\begin{aligned} & \int_{\mathbb{R}^2} \nabla f \nabla \tilde{\psi}_1 dy = 8 \int_{\mathbb{R}^2} \frac{f \tilde{\psi}_1}{(1 + |y|^2)^2} dy \\ & - \frac{4}{\rho} \int_{\mathbb{R}^2} \frac{\tilde{\psi}_1}{(1 + |y|^2)^2} dy \left(8 \int_{\mathbb{R}^2} \frac{f}{(1 + |y|^2)^2} dy + 2 \frac{\rho - 4\pi}{\int_{\Omega} e^z} \int_{\Omega} e^z g dx \right). \end{aligned}$$

Thus, since a single point has capacity zero in \mathbb{R}^2 , we deduce that the function

$$f - \frac{4}{\rho} \int_{\mathbb{R}^2} \frac{f}{(1 + |y|^2)^2} dy - \frac{\rho - 4\pi}{\rho \int_{\Omega} e^z} \int_{\Omega} e^z g dx \in H_2(\mathbb{R}^2)$$

is a solution of the equation

$$-\Delta\phi_0 = \frac{8}{(1+|y|^2)^2}\phi_0 \quad \text{in } \mathbb{R}^2.$$

Recall now that f satisfies (2.1) and, by Proposition 2.5,

$$f - \frac{4}{\rho} \int_{\mathbb{R}^2} \frac{f}{(1+|y|^2)^2} dy - \frac{\rho-4\pi}{\rho \int_{\Omega} e^z} \int_{\Omega} e^z g dx = \gamma_1 \frac{1-|y|^2}{1+|y|^2} \quad (4.16)$$

for some $\gamma_1 \in \mathbb{R}$. Since $\langle 1, \frac{1-|y|^2}{1+|y|^2} \rangle_{L_2} = \int_{\mathbb{R}^2} \frac{1-|y|^2}{(1+|y|^2)^3} = 0$, then

$$\frac{4}{\rho} \int_{\mathbb{R}^2} \frac{f}{(1+|y|^2)^2} dy + \frac{\rho-4\pi}{\rho \int_{\Omega} e^z} \int_{\Omega} e^z g dx$$

actually coincides with the projection of f onto the space of constants, namely,

$$\frac{1}{\pi} \int_{\mathbb{R}^2} \frac{f}{(1+|y|^2)^2} dy = \frac{4}{\rho} \int_{\mathbb{R}^2} \frac{f}{(1+|y|^2)^2} dy + \frac{\rho-4\pi}{\rho \int_{\Omega} e^z} \int_{\Omega} e^z g dx,$$

by which

$$\frac{1}{\pi} \int_{\mathbb{R}^2} \frac{f}{(1+|y|^2)^2} dy = \frac{1}{\int_{\Omega} e^z} \int_{\Omega} e^z g dx, \quad (4.17)$$

By inserting this into (4.14) we get

$$\frac{1}{\int_{\Omega} e^{W_1}} \int_{\Omega} e^{W_1} \phi_{1n} dx = \frac{1}{\int_{\Omega} e^z} \int_{\Omega} e^z g dx + o(1). \quad (4.18)$$

Next let us fix $\psi_1 \in C_c^\infty(\Omega \setminus \{0\})$; so, we multiply the first equation in (4.1) by ψ_1 , we integrate over Ω and we get

$$\begin{aligned} & \int_{\Omega} \nabla \phi_{1n} \nabla \psi_1 dx - 2\rho \left[\frac{\int_{\Omega} e^{W_1} \phi_{1n} \psi_1 dx}{\int_{\Omega} e^{W_1}} - \frac{\int_{\Omega} e^{W_1} \psi_1 dx \int_{\Omega} e^{W_1} \phi_{1n} dx}{(\int_{\Omega} e^{W_1})^2} \right] \\ & + \lambda_n \int_{\Omega} e^{W_2} \phi_{2n} \psi_1 dx = - \int_{\Omega} \nabla h_{1n} \nabla \psi_1. \end{aligned}$$

By (4.7), and recalling (3.4) and (4.18), we pass to the limit to obtain

$$\int_{\Omega} \nabla g \nabla \psi_1 dx - 2(\rho-4\pi) \left[\frac{\int_{\Omega} e^z g \psi_1 dx}{\int_{\Omega} e^z} - \frac{\int_{\Omega} e^z \psi_1 dx \int_{\Omega} e^z g dx}{(\int_{\Omega} e^z)^2} \right] = 0.$$

Since a single point has capacity zero in \mathbb{R}^2 , then the above identity turns out to hold for every $\psi_1 \in C_c^\infty(\Omega)$; we deduce that $g \in H_0^1(\Omega)$ solves the problem

$$-\Delta g - 2(\rho-4\pi) \left[\frac{e^z g}{\int_{\Omega} e^z} - \frac{e^z \int_{\Omega} e^z g dx}{(\int_{\Omega} e^z)^2} \right] = 0 \quad \text{in } \Omega.$$

Therefore, by (H) we get $g = 0$. (4.17) gives $\int_{\mathbb{R}^2} \frac{f}{(1+|y|^2)^2} = 0$ and, finally, by (4.16) we conclude $f = \gamma_1 \frac{1-|y|^2}{1+|y|^2}$. We have thus proved (4.10) and (4.11).

Observe that, in particular,

$$\int_{\Omega} e^{w_1} \phi_{1n} dx = 8 \langle \tilde{\phi}_{1n}, 1 \rangle_{L_2} = o(1), \quad \int_{\Omega} e^z \phi_{1n} dx = o(1). \quad (4.19)$$

In order to prove (4.12), observe that Step 1 and Proposition 2.4 give

$$\tilde{\phi}_{2n} \rightarrow h \text{ weakly in } H_4(\mathbb{R}^2) \text{ and strongly in } L_4(\mathbb{R}^2).$$

Since $\tilde{\phi}_{2n}$ satisfies (2.1), then h also satisfies (2.1). Let us fix $\tilde{\psi}_2 \in C_c^\infty(\mathbb{R}^2 \setminus \{0\})$ and set $\psi_{2n} = \tilde{\psi}_2(\frac{x}{\delta_{2n}})$, $x \in \Omega$. Then $\psi_{2n} \in C_c^\infty(\Omega \setminus \{0\})$; so, we multiply the second equation in (4.1) by ψ_{2n} , we integrate over Ω and we get

$$\begin{aligned} \int_{\tilde{\Omega}_{2n}} \nabla \tilde{\phi}_{2n} \nabla \tilde{\psi}_2 dy + \rho \left[\frac{\int_{\Omega} e^{W_1} \phi_{1n} \psi_{2n} dx}{\int_{\Omega} e^{W_1} dx} - \frac{\int_{\Omega} e^{W_1} \psi_{2n} dx \int_{\Omega} e^{W_1} \phi_{1n} dx}{(\int_{\Omega} e^{W_1} dx)^2} \right] \\ - 2\lambda_n \int_{\Omega} e^{W_2} \phi_{2n} \psi_{2n} dx = - \int_{\Omega} \nabla h_{2n} \nabla \psi_{2n} dx \end{aligned} \quad (4.20)$$

Since $e^{w_1} \leq C$ in the support of ψ_{2n} , then Lemma 3.1 gives $\frac{1}{\int_{\Omega} e^{W_1}} e^{W_1} \psi_{2n} \rightarrow 0$ in $L^q(\Omega)$ for all $q \in (1, 2)$, and so

$$\frac{1}{\int_{\Omega} e^{W_1}} \int_{\Omega} e^{W_1} \psi_{2n} dx = o(1), \quad \frac{1}{\int_{\Omega} e^{W_1}} \int_{\Omega} e^{W_1} \phi_{1n} \psi_{2n} dx = o(1).$$

Again by Lemma 3.1 we compute

$$2\lambda_n \int_{\Omega} e^{W_2} \phi_{2n} \psi_{2n} dx = \int_{\Omega} |x|^2 e^{w_2} \phi_{2n} \psi_{2n} dx + o(1) = 32 \int_{\mathbb{R}^2} \frac{|y|^4}{(1 + |y|^4)^2} h \tilde{\psi}_2 dy + o(1).$$

Similarly to (4.15), we have $\int_{\Omega} \nabla h_{2n} \nabla \psi_{2n} = o(1)$. Then, recalling (4.18) we can pass to the limit in (4.20) to obtain

$$\int_{\mathbb{R}^2} \nabla h \nabla \tilde{\psi}_2 dy - 32 \int_{\mathbb{R}^2} \frac{|y|^4}{(1 + |y|^4)^2} h \tilde{\psi}_2 dy = 0. \quad (4.21)$$

The above identity turns out to hold for every $\tilde{\psi}_2 \in C_c^\infty(\mathbb{R}^2)$; we deduce that $h \in H_4(\mathbb{R}^2)$ solves the problem

$$-\Delta h - 32 \frac{|y|^4}{(1 + |y|^4)^2} h = 0 \text{ in } \Omega.$$

Recalling that h satisfies (2.1), Proposition 2.5 implies $h = 0$. This proves (4.12).

We point out that

$$\int_{\Omega} |x|^2 e^{w_2} \phi_{2n} dx = 32 \langle \tilde{\phi}_{2n}, 1 \rangle_{L_4} = o(1). \quad (4.22)$$

Step 3. We will show that

$$\begin{aligned} \langle \tilde{\phi}_{1n}, 1 \rangle_{L_2} - \frac{\pi}{\int_{\Omega} e^z} \int_{\Omega} e^z \phi_{1n} dx &= o\left(\frac{1}{|\log \lambda_n|}\right), \\ \langle \tilde{\phi}_{2n}, 1 \rangle_{L_4} &= o\left(\frac{1}{|\log \lambda_n|}\right). \end{aligned}$$

It is important to notice that, by (4.19) and (4.22), both expressions converge to 0. This step is devoted to prove an estimate on the speed of convergence, which will be crucial for step 4.

Let $Z_1(x) = \frac{\delta_1^2 - |x|^2}{\delta_1^2 + |x|^2}$ $Z_2(x) = \frac{\delta_2^4 - |x|^4}{\delta_2^4 + |x|^4}$ be the radial solution to the linear problems

$$-\Delta Z_1 = 8 \frac{\delta_1^2}{(\delta_1^2 + |x|^2)^2} Z_1 \text{ in } \mathbb{R}^2, \quad -\Delta Z_2 = 32 \frac{\delta_2^4 |x|^2}{(\delta_2^4 + |x|^4)^2} Z_2 \text{ in } \mathbb{R}^2.$$

Let PZ_1, PZ_2 be their projection onto $H_0^1(\Omega)$. By the maximum principle it is not difficult to check that

$$\begin{aligned} PZ_1 &= Z_1 + 1 + O(\delta_1^2) = \frac{2\delta_1^2}{\delta_1^2 + |x|^2} + O(\lambda), \\ PZ_2 &= Z_2 + 1 + O(\delta_2^4) = \frac{2\delta_2^4}{\delta_2^4 + |x|^4} + O(\lambda). \end{aligned} \quad (4.23)$$

We observe that

$$\|PZ_1\|_q^q = O(\delta_1^2), \quad \|PZ_2\|_q^q = O(\delta_2^2) \quad \forall q > 1. \quad (4.24)$$

Now, we multiply the first equation in (4.1) by PZ_1 , we integrate over Ω and we get

$$\begin{aligned} & \int_{\Omega} \nabla \phi_{1n} \nabla PZ_1 dx - 2\rho \left[\frac{\int_{\Omega} e^{W_1} \phi_{1n} PZ_1 dx}{\int_{\Omega} e^{W_1} dx} - \frac{\int_{\Omega} e^{W_1} PZ_1 dx \int_{\Omega} e^{W_1} \phi_{1n} dx}{(\int_{\Omega} e^{W_1} dx)^2} \right] \\ & + \lambda_n \int_{\Omega} e^{W_2} \phi_{2n} PZ_1 dx = - \int_{\Omega} \nabla h_{1n} \nabla PZ_1 dx. \end{aligned} \quad (4.25)$$

We are now concerned with the estimates of each term of the above expression.

Since, for any $q \geq 1$,

$$\begin{aligned} \| |x|^2 e^{w_2} PZ_1 \|_q &= \left\| |x|^2 e^{w_2} \frac{2\delta_{1n}^2}{\delta_{1n}^2 + |x|^2} \right\|_q + O(\lambda_n) \| |x|^2 e^{w_2} \|_q \\ &\leq 2\delta_{1n}^2 \| e^{w_2} \|_q + O(\lambda_n) \| |x|^2 e^{w_2} \|_q = O(\lambda_n^{\frac{1}{2q}}) \end{aligned} \quad (4.26)$$

by Lemma 3.1 we conclude:

$$2\lambda_n \int_{\Omega} e^{W_2} \phi_{2n} PZ_1 dx = \int_{\Omega} |x|^2 e^{w_2} \phi_{2n} PZ_1 dx + o\left(\frac{1}{|\log \lambda_n|}\right) = o\left(\frac{1}{|\log \lambda_n|}\right). \quad (4.27)$$

Next, we compute

$$\int_{\Omega} \nabla \phi_{1n} \nabla PZ_1 dx = 8 \int_{\Omega} \phi_{1n} \frac{\delta_{1n}^2}{(\delta_{1n}^2 + |x|^2)^2} Z_1 dx = \int_{\Omega} e^{w_1} \phi_{1n} Z_1 dx. \quad (4.28)$$

According to Step 1 we have

$$\int_{\Omega} |e^{w_1} \phi_{1n}| dx = 8 \langle |\tilde{\phi}_{1n}|, 1 \rangle_{L_2} = O(1). \quad (4.29)$$

By Lemma 3.1, using (4.24) and (4.29),

$$\begin{aligned}
\frac{2\rho}{\int_{\Omega} e^{W_1}} \int_{\Omega} e^{W_1} \phi_{1n} P Z_1 dx &= \int_{\Omega} e^{w_1} \phi_{1n} P Z_1 dx + 2 \frac{\rho - 4\pi}{\int_{\Omega} e^z} \int_{\Omega} e^z \phi_{1n} P Z_1 dx + o\left(\frac{1}{|\log \lambda_n|}\right) \\
&= \int_{\Omega} e^{w_1} \phi_{1n} (Z_1 + 1) dx + o\left(\frac{1}{|\log \lambda_n|}\right). \\
&= \int_{\Omega} e^{w_1} \phi_{1n} Z_1 dx + 8 \langle \tilde{\phi}_{1n}, 1 \rangle_{L_2} + o\left(\frac{1}{|\log \lambda_n|}\right).
\end{aligned} \tag{4.30}$$

Similarly, by using again Lemma 3.1 and (4.24)

$$\begin{aligned}
\frac{2\rho}{\int_{\Omega} e^{W_1}} \int_{\Omega} e^{W_1} P Z_1 dx &= \int_{\Omega} e^{w_1} (Z_1 + 1) dx + o\left(\frac{1}{|\log \lambda_n|}\right) \\
&= \int_{\Omega} e^{w_1} \frac{2\delta_{1n}^2}{\delta_{1n}^2 + |x|^2} dx + o\left(\frac{1}{|\log \lambda_n|}\right) = 8\pi + o\left(\frac{1}{|\log \lambda_n|}\right)
\end{aligned} \tag{4.31}$$

since

$$\int_{\Omega} e^{w_1} \frac{\delta_{1n}^2}{\delta_{1n}^2 + |x|^2} dx = 8 \int_{\widetilde{\Omega}_{1n}} \frac{dy}{(1 + |y|^2)^3} = 8 \int_{\mathbb{R}^2} \frac{dy}{(1 + |y|^2)^3} + O(\delta_{1n}^2) = 4\pi + O(\delta_{1n}^2).$$

Moreover, by Lemma 3.1,

$$\begin{aligned}
\frac{2\rho}{\int_{\Omega} e^{W_1}} \int_{\Omega} e^{W_1} \phi_{1n} dx &= \int_{\Omega} e^{w_1} \phi_{1n} dx + 2 \frac{\rho - 4\pi}{\int_{\Omega} e^z} \int_{\Omega} e^z \phi_{1n} dx + o\left(\frac{1}{|\log \lambda_n|}\right) \\
&= 8 \langle \tilde{\phi}_{1n}, 1 \rangle_{L_2} + 2 \frac{\rho - 4\pi}{\int_{\Omega} e^z} \int_{\Omega} e^z \phi_{1n} dx + o\left(\frac{1}{|\log \lambda_n|}\right).
\end{aligned} \tag{4.32}$$

Finally, since $P Z_1 = O(1)$, we have $\int_{\Omega} |\nabla P Z_1|^2 = \int_{\Omega} e^{w_1} P Z_1 = O(\int_{\Omega} e^{w_1}) = O(1)$, by which, owing to (4.3),

$$\int_{\Omega} |\nabla h_{1n} \nabla P Z_1| dx \leq \|h_{1n}\| \|P Z_1\| = \|h_{1n}\| = o\left(\frac{1}{|\log \lambda_n|}\right). \tag{4.33}$$

We now multiply (4.25) by $\log \lambda_n$ and pass to the limit, inserting (4.27), (4.28), (4.30), (4.31), (4.32) and (4.33), to obtain

$$\log \lambda_n \langle \tilde{\phi}_{1n}, 1 \rangle_{L_2} - \log \lambda_n \frac{\pi}{\int_{\Omega} e^z} \int_{\Omega} e^z \phi_{1n} dx = o(1) \tag{4.34}$$

and the first part of the thesis follows.

For the second part, we multiply the second equation in (4.1) by $P Z_2$ and integrate, to obtain:

$$\begin{aligned} & \int_{\Omega} \nabla \phi_{2n} \nabla P Z_2 dx + \rho \left[\frac{\int_{\Omega} e^{W_1} \phi_{1n} P Z_2 dx}{\int_{\Omega} e^{W_1} dx} - \frac{\int_{\Omega} e^{W_1} P Z_2 dx \int_{\Omega} e^{W_1} \phi_{1n} dx}{\left(\int_{\Omega} e^{W_1} dx\right)^2} \right] \\ & - 2\lambda_n \int_{\Omega} e^{W_2} \phi_{2n} P Z_2 dx = - \int_{\Omega} \nabla h_{2n} \nabla P Z_2 dx. \end{aligned} \quad (4.35)$$

We now estimate each of the terms above. Observe that:

$$\int_{\Omega} \nabla \phi_{2n} \nabla P Z_2 dx = 32 \int_{\Omega} \phi_{2n} \frac{\delta_{2n}^4 |x|^2}{(\delta_{2n}^4 + |x|^4)^2} Z_2 dx = \int_{\Omega} |x|^2 e^{w_2} \phi_{2n} Z_2 dx. \quad (4.36)$$

According to Step 1,

$$\int_{\Omega} ||x|^2 e^{w_2} \phi_{2n}| = 32 \langle |\tilde{\phi}_{2n}|, 1 \rangle_{L_4} = O(1). \quad (4.37)$$

By Lemma 3.1, (4.24) and (4.37) we compute

$$\begin{aligned} 2\lambda_n \int_{\Omega} e^{W_2} \phi_{2n} P Z_2 dx &= \int_{\Omega} |x|^2 e^{w_2} \phi_{2n} P Z_2 dx + o\left(\frac{1}{|\log \lambda_n|}\right) \\ &= \int_{\Omega} |x|^2 e^{w_2} \phi_{2n} (Z_2 + 1) dx + o\left(\frac{1}{|\log \lambda_n|}\right) \\ &= \int_{\Omega} |x|^2 e^{w_2} \phi_{2n} Z_2 dx + 32 \langle \tilde{\phi}_{2n}, 1 \rangle_{L_4} + o\left(\frac{1}{|\log \lambda_n|}\right). \end{aligned} \quad (4.38)$$

Next, by using Lemma 3.1, recalling (4.24) and (4.29),

$$\begin{aligned} \frac{2\rho}{\int_{\Omega} e^{W_1}} \int_{\Omega} e^{W_1} \phi_{1n} P Z_2 dx &= \int_{\Omega} e^{w_1} \phi_{1n} P Z_2 dx + 2 \frac{\rho - 4\pi}{\int_{\Omega} e^z} \int_{\Omega} e^z P Z_2 dx + o\left(\frac{1}{|\log \lambda_n|}\right) \\ &= \int_{\Omega} e^{w_1} \phi_{1n} \frac{2\delta_{2n}^4}{\delta_{2n}^4 + |x|^4} dx + o\left(\frac{1}{|\log \lambda_n|}\right). \end{aligned} \quad (4.39)$$

Let us observe that, using Step 1, and denoting by χ_A the characteristic function of the set A ,

$$\begin{aligned} \int_{|x| \geq \delta_{2n}^{3/2}} e^{w_1} |\phi_{1n}| dx &= 8 \langle \chi_{\{|y| \geq \frac{\delta_{2n}^{3/2}}{\delta_{1n}}\}}, \tilde{\phi}_{1n} \rangle_{L_2} \leq C \|\chi_{\{|y| \geq \frac{\delta_{2n}^{3/2}}{\delta_{1n}}\}}\|_{L_2} \\ &= C \left(\int_{|y| \geq \frac{\delta_{2n}^{3/2}}{\delta_{1n}}} \frac{1}{(1 + |y|^2)^2} dy \right)^{1/2} \leq C \frac{\delta_{1n}^{1/2}}{\delta_{2n}^{3/4}} \end{aligned}$$

by which (4.39) becomes

$$\frac{2\rho}{\int_{\Omega} e^{W_1}} \int_{\Omega} e^{W_1} \phi_{1n} P Z_2 dx = \int_{|x| \leq \delta_{2n}^{3/2}} e^{w_1} \phi_{1n} \frac{2\delta_{2n}^4}{\delta_{2n}^4 + |x|^4} dx + o\left(\frac{1}{|\log \lambda_n|}\right)$$

and, since $\frac{2\delta_{2n}^4}{\delta_{2n}^4 + |x|^4} = 2 + O(\delta_{2n}^2)$ for $|x| \leq \delta_{2n}^{3/2}$,

$$\begin{aligned} \frac{2\rho}{\int_{\Omega} e^{W_1}} \int_{\Omega} e^{W_1} \phi_{1n} P Z_2 dx &= 2 \int_{|x| \leq \delta_{2n}^{3/2}} e^{w_1} \phi_{1n} dx + o\left(\frac{1}{|\log \lambda_n|}\right) \\ &= 2 \int_{\mathbb{R}^2} e^{w_1} \phi_{1n} dx + o\left(\frac{1}{|\log \lambda_n|}\right) \\ &= 16 \langle \tilde{\phi}_{1n}, 1 \rangle_{L_2} + o\left(\frac{1}{|\log \lambda_n|}\right). \end{aligned} \quad (4.40)$$

The identical computation hold by replacing ϕ_{1n} by 1 in (4.40) and so

$$\frac{2\rho}{\int_{\Omega} e^{W_1}} \int_{\Omega} e^{W_1} P Z_2 dx = 16 \|1\|_{L_2}^2 + o\left(\frac{1}{|\log \lambda_n|}\right) = 16\pi + o\left(\frac{1}{|\log \lambda_n|}\right). \quad (4.41)$$

Finally, similarly to (4.31),

$$\int_{\Omega} \nabla h_{1n} \nabla P Z_1 dx = o\left(\frac{1}{|\log \lambda_n|}\right) \quad (4.42)$$

By multiplying (4.35) by $\log \lambda_n$, passing to the limit, and inserting (4.36), (4.38), (4.40), (4.41) and (4.42), and recalling (4.32), we arrive at

$$-\log \lambda_n \langle \tilde{\phi}_{2n}, 1 \rangle_{L_4} + \frac{\rho - 4\pi}{4\rho} \log \lambda_n \left(\langle \tilde{\phi}_{1n}, 1 \rangle_{L_2} - \frac{\pi}{\int_{\Omega} e^z} \int_{\Omega} e^z \phi_{1n} dx \right) = o(1). \quad (4.43)$$

Combining (4.34) with (4.43) we obtain the thesis.

Before going on, we recall the following identities which follow by straightforward computations: for every $\alpha \geq 2$

$$\left\langle \frac{1 - |y|^\alpha}{1 + |y|^\alpha}, 1 \right\rangle_{L_\alpha} = \int_{\mathbb{R}^2} \frac{|y|^{\alpha-2}}{(1 + |y|^\alpha)^2} \frac{1 - |y|^\alpha}{1 + |y|^\alpha} dy = 0, \quad (4.44)$$

$$\left\langle \frac{1 - |y|^\alpha}{1 + |y|^\alpha}, \log(1 + |y|^\alpha) \right\rangle_{L_\alpha} = \int_{\mathbb{R}^2} \frac{|y|^{\alpha-2}}{(1 + |y|^\alpha)^2} \frac{1 - |y|^\alpha}{1 + |y|^\alpha} \log(1 + |y|^\alpha) dy = -\frac{\pi}{\alpha}, \quad (4.45)$$

$$\left\langle \frac{1 - |y|^\alpha}{1 + |y|^\alpha}, \log |y| \right\rangle_{L_\alpha} = \int_{\mathbb{R}^2} \frac{|y|^{\alpha-2}}{(1 + |y|^\alpha)^2} \frac{1 - |y|^\alpha}{1 + |y|^\alpha} \log |y| dy = -\frac{\pi}{2\alpha^2}. \quad (4.46)$$

Step 4. We will show that $\gamma_1 = \gamma_2 = 0$.

We first multiply the first equation in (4.1) by $P w_1$ and integrate; we obtain:

$$\begin{aligned} \int_{\Omega} \nabla \phi_{1n} \nabla P w_1 dx - 2\rho \left[\frac{\int_{\Omega} e^{W_1} \phi_{1n} P w_1 dx}{\int_{\Omega} e^{W_1} dx} - \frac{\int_{\Omega} e^{W_1} P w_1 dx \int_{\Omega} e^{W_1} \phi_{1n} dx}{\left(\int_{\Omega} e^{W_1} dx\right)^2} \right] \\ + \lambda_n \int_{\Omega} e^{W_2} \phi_{2n} P w_1 dx = - \int_{\Omega} \nabla h_{1n} \nabla P w_1 dx. \end{aligned} \quad (4.47)$$

Let us estimate each of the terms above. By (4.19),

$$\int_{\Omega} \nabla \phi_{1n} \nabla Pw_1 dx = \int_{\Omega} e^{w_1} \phi_{1n} dx = o(1). \quad (4.48)$$

By Lemma 3.1, using that $|Pw_1| = O(|\log \lambda_n|)$, we get

$$\begin{aligned} \frac{2\rho}{\int_{\Omega} e^{W_1}} \int_{\Omega} e^{W_1} \phi_{1n} Pw_1 dx &= \int_{\Omega} e^{w_1} \phi_{1n} Pw_1 dx + 2 \frac{\rho - 4\pi}{\int_{\Omega} e^z} \int_{\Omega} e^z \phi_{1n} Pw_1 dx + o(1) \\ &= 8 \langle \tilde{\phi}_{1n}, Pw_1(\delta_{1n}y) \rangle_{L_2} + 2 \frac{\rho - 4\pi}{\int_{\Omega} e^z} \int_{\Omega} e^z \phi_{1n} Pw_1 dx + o(1). \end{aligned} \quad (4.49)$$

Observe that by (2.8)

$$Pw_1 \rightarrow -2 \log |x|^2 + 8\pi H(x, 0) \quad \text{in } L^q(\Omega) \quad \forall q \geq 1. \quad (4.50)$$

Moreover

$$Pw_1(\delta_{1n}y) = -2 \log(1 + |y|^2) + 8\pi H(\delta_{1n}y, 0) - 4 \log \delta_{1n} + O(\lambda_n),$$

by which

$$Pw_1(\delta_{1n}y) + 4 \log \delta_{1n} \rightarrow -2 \log(1 + |y|^2) + 8\pi H(0, 0) \quad \text{in } L_2(\mathbb{R}^2). \quad (4.51)$$

Using these convergences into (4.49), and recalling Step 3, we obtain

$$\begin{aligned} \frac{2\rho}{\int_{\Omega} e^{W_1}} \int_{\Omega} e^{W_1} \phi_{1n} Pw_1 dx &= -16\gamma_1 \left\langle \frac{1 - |y|^2}{1 + |y|^2}, \log(1 + |y|^2) \right\rangle_{L_2} \\ &\quad + 64\pi H(0, 0) \gamma_1 \left\langle \frac{1 - |y|^2}{1 + |y|^2}, 1 \right\rangle_{L_2} - 32 \log \delta_{1n} \langle \tilde{\phi}_{1n}, 1 \rangle_{L_2} + o(1) \\ &= 8\pi\gamma_1 - 32 \log \delta_{1n} \langle \tilde{\phi}_{1n}, 1 \rangle_{L_2} + o(1) \end{aligned} \quad (4.52)$$

by (4.44) and (4.45).

Proceeding similarly as in (4.49) with 1 in the place of ϕ_{1n} we deduce

$$\frac{2\rho}{\int_{\Omega} e^{W_1}} \int_{\Omega} e^{W_1} Pw_1 dx = 8 \langle 1, Pw_1(\delta_{1n}y) \rangle_{L_2} + 2 \frac{\rho - 4\pi}{\int_{\Omega} e^z} \int_{\Omega} e^z Pw_1 dx + o(1) \quad (4.53)$$

and using (4.50)-(4.51)

$$\begin{aligned} \frac{2\rho}{\int_{\Omega} e^{W_1}} \int_{\Omega} e^{W_1} Pw_1 dx &= -32 \log \delta_{1n} \|1\|_{L_2}^2 - 16 \langle 1, \log(1 + |y|^2) \rangle_{L_2} + 64\pi H(0, 0) \|1\|_{L_2}^2 \\ &\quad + O(1) \\ &= -32\pi \log \delta_{1n} + O(1) \end{aligned} \quad (4.54)$$

by $\|1\|_{L_2}^2 = \pi$. Combining (4.32) with (4.54) and taking into account (4.19) we have

$$\begin{aligned} \frac{2\rho}{(\int_{\Omega} e^{W_1})^2} \int_{\Omega} e^{W_1} Pw_1 dx \int_{\Omega} e^{W_1} \phi_{1n} dx &= -\frac{128}{\rho} \pi \log \delta_{1n} \langle \tilde{\phi}_{1n}, 1 \rangle_{L_2} \\ &\quad - \frac{32}{\rho} \pi \log \delta_{1n} \frac{\rho - 4\pi}{\int_{\Omega} e^z} \int_{\Omega} e^z \phi_{1n} dx + o(1). \end{aligned} \quad (4.55)$$

Next we use Lemma 3.1 and that $|Pw_1| = O(|\log \lambda_n|)$ and we get

$$\begin{aligned} 2\lambda_n \int_{\Omega} e^{W_2} \phi_{2n} Pw_1 dx &= \int_{\Omega} |x|^2 e^{w_2} \phi_{2n} Pw_1 dx + o(1) \\ &= 32 \langle \tilde{\phi}_{2n}, Pw_1(\delta_{2n} y) \rangle_{L_4} + o(1). \end{aligned} \quad (4.56)$$

Observe that by (2.8)

$$\begin{aligned} Pw_1(\delta_{2n} y) &= -2 \log(\delta_{1n}^2 + \delta_{2n}^2 |y|^2) + 8\pi H(\delta_{2n} y, 0) + O(\lambda) \\ &= -4 \log \delta_{2n} - 2 \log \left(\left(\frac{\delta_{1n}}{\delta_{2n}} \right)^2 + |y|^2 \right) + 8\pi H(\delta_{2n} y, 0) + O(\lambda_n) \end{aligned}$$

by which

$$Pw_1(\delta_{2n} y) + 4 \log \delta_{2n} \rightarrow -4 \log |y| + 8\pi H(0, 0) \quad \text{in } L_4(\mathbb{R}^2).$$

By inserting this convergence in (4.56) we deduce

$$\begin{aligned} 2\lambda_n \int_{\Omega} e^{W_2} \phi_{2n} Pw_1 dx &= -128 \log \delta_{2n} \langle \tilde{\phi}_{2n}, 1 \rangle_{L_4} - 128 \gamma_2 \left\langle \frac{1 - |y|^4}{1 + |y|^4}, \log |y| \right\rangle_{L_4} \\ &\quad + 256 \pi H(0, 0) \gamma_2 \left\langle \frac{1 - |y|^4}{1 + |y|^4}, 1 \right\rangle_{L_4} + o(1) \\ &= 4\pi \gamma_2 + o(1) \end{aligned} \quad (4.57)$$

where we have used (4.44), (4.46) and Step 3.

Finally, since $|Pw_1| = O(|\log \lambda_n|)$, we have $\int_{\Omega} |\nabla Pw_1|^2 = \int_{\Omega} e^{w_1} Pw_1 = O(|\log \lambda_n| \int_{\Omega} e^{w_1}) = O(|\log \lambda_n|)$ and so, by (4.3),

$$\int_{\Omega} |\nabla h_{1n} \nabla Pw_1| \leq \|h_{1n}\| \|Pw_1\| = o(1). \quad (4.58)$$

Passing to the limit in (4.47) and using (4.48), (4.52), (4.55), (4.57) and (4.58)

$$-8\pi \gamma_1 + \frac{32\rho - 128\pi}{\rho} \log \delta_{1n} \langle \tilde{\phi}_{1n}, 1 \rangle_{L_2} - \frac{32}{\rho} \pi \log \delta_{1n} \frac{\rho - 4\pi}{\int_{\Omega} e^z} \int_{\Omega} e^z \phi_{1n} + 2\pi \gamma_2 = o(1)$$

and then, by Step 3,

$$4\gamma_1 - \gamma_2 = o(1). \quad (4.59)$$

Next, we multiply the second equation in (4.1) by Pw_2 , we integrate over Ω and we get

$$\begin{aligned} \int_{\Omega} \nabla \phi_{2n} \nabla Pw_2 dx + \rho \left[\frac{\int_{\Omega} e^{W_1} \phi_{1n} Pw_2 dx}{\int_{\Omega} e^{W_1} dx} - \frac{\int_{\Omega} e^{W_1} Pw_2 dx \int_{\Omega} e^{W_1} \phi_{1n} dx}{\left(\int_{\Omega} e^{W_1} dx \right)^2} \right] \\ - 2\lambda_n \int_{\Omega} e^{W_2} \phi_{2n} Pw_2 dx = - \int_{\Omega} \nabla h_{1n} \nabla Pw_1 dx. \end{aligned} \quad (4.60)$$

Again, we estimate each of the terms above. By (4.22),

$$\int_{\Omega} \nabla \phi_{2n} \nabla Pw_2 dx = \int_{\Omega} |x|^2 e^{w_2} \phi_{2n} dx = o(1). \quad (4.61)$$

By Lemma 3.1, using that $|Pw_2| = O(|\log \lambda_n|)$, we get

$$2\lambda_n \int_{\Omega} e^{W_2} \phi_{2n} Pw_2 dx = \int_{\Omega} |x|^2 e^{w_2} \phi_{2n} Pw_2 dx + o(1) = 32 \langle \tilde{\phi}_{2n}, Pw_2(\delta_{2n}y) \rangle_{L_4} + o(1). \quad (4.62)$$

Observe that by (2.8)

$$Pw_2(\delta_{2n}y) = -2 \log(1 + |y|^4) + 16\pi H(\delta_{2n}y, 0) - 8 \log \delta_{2n} + O(\lambda_n),$$

by which

$$Pw_2(\delta_{2n}y) + 8 \log \delta_{2n} \rightarrow -2 \log(1 + |y|^4) + 16\pi H(0, 0) \quad \text{in } L_4(\mathbb{R}^2). \quad (4.63)$$

Using these convergences into (4.62), and recalling Step 3, we obtain

$$\begin{aligned} 2\lambda_n \int_{\Omega} e^{W_2} \phi_{2n} Pw_2 dx &= -64\gamma_2 \left\langle \frac{1 - |y|^4}{1 + |y|^4}, \log(1 + |y|^4) \right\rangle_{L_4} + 512\pi H(0, 0)\gamma_2 \left\langle \frac{1 - |y|^4}{1 + |y|^4}, 1 \right\rangle_{L_4} \\ &\quad - 256 \log \delta_{2n} \langle \tilde{\phi}_{2n}, 1 \rangle_{L_4} + o(1) \\ &= 16\pi\gamma_2 + o(1) \end{aligned} \quad (4.64)$$

by (4.44) and (4.45). Again by Lemma 3.1, taking into account that $|Pw_2| = O(|\log \lambda_n|)$, we get

$$\begin{aligned} \frac{2\rho}{\int_{\Omega} e^{W_1}} \int_{\Omega} e^{W_1} \phi_{1n} Pw_2 dx &= \int_{\Omega} e^{w_1} \phi_{1n} Pw_2 dx + 2 \frac{\rho - 4\pi}{\int_{\Omega} e^z} \int_{\Omega} e^z \phi_{1n} Pw_2 dx + o(1) \\ &= 8 \langle \tilde{\phi}_{1n}, Pw_2(\delta_{1n}y) \rangle_{L_2} + 2 \frac{\rho - 4\pi}{\int_{\Omega} e^z} \int_{\Omega} e^z \phi_{1n} Pw_2 dx + o(1) \end{aligned} \quad (4.65)$$

Again by (2.8)

$$Pw_2 \rightarrow -8 \log |y| + 16\pi H(x, 0) \quad \text{in } L^q(\Omega) \quad \forall q \geq 1.$$

Moreover

$$\begin{aligned} Pw_2(\delta_{1n}y) &= -2 \log(\delta_{2n}^4 + \delta_{1n}^4 |y|^4) + 16\pi H(\delta_{1n}y, 0) + O(\lambda_n) \\ &= -8 \log \delta_{2n} - 2 \log \left(1 + \left(\frac{\delta_{1n}}{\delta_{2n}} \right)^4 |y|^4 \right) + 16\pi H(\delta_{1n}y, 0) + O(\lambda_n) \end{aligned}$$

by which

$$Pw_2(\delta_{1n}y) + 8 \log \delta_{2n} \rightarrow 16\pi H(0, 0) \quad \text{in } L_2(\mathbb{R}^2).$$

By inserting these convergences into (4.65) we obtain

$$\begin{aligned} \frac{2\rho}{\int_{\Omega} e^{W_1}} \int_{\Omega} e^{W_1} \phi_{1n} Pw_2 dx &= -64 \log \delta_{2n} \langle \tilde{\phi}_{1n}, 1 \rangle_{L_2} + 128 H(0, 0) \pi \gamma_1 \left\langle \frac{1 - |y|^2}{1 + |y|^2}, 1 \right\rangle_{L_2} \\ &\quad + 2 \frac{\rho - 4\pi}{\int_{\Omega} e^z} \int_{\Omega} e^z \phi_{1n} (-8 \log |y| + 16\pi H(x, 0)) dx + o(1) \\ &= -64 \log \delta_{2n} \langle \tilde{\phi}_{1n}, 1 \rangle_{L_2} + o(1). \end{aligned} \quad (4.66)$$

Similarly, by replacing ϕ_{1n} by 1 in (4.65),

$$\begin{aligned} \frac{2\rho}{\int_{\Omega} e^{W_1}} \int_{\Omega} e^{W_1} P w_2 dx &= 8 \langle 1, P w_2(\delta_{1n} y) \rangle_{L_2} + 2 \frac{\rho - 4\pi}{\int_{\Omega} e^z} \int_{\Omega} e^z P w_2 dx + o(1) \\ &= -64 \log \delta_{2n} \|1\|_{L_2}^2 + 128\pi H(0, 0) \|1\|_{L_2}^2 \\ &\quad + 2 \frac{\rho - 4\pi}{\int_{\Omega} e^z} \int_{\Omega} e^z (-8 \log |y| + 16\pi H(x, 0)) dx + o(1) \\ &= -64\pi \log \delta_{2n} + O(1). \end{aligned}$$

Combining this with (4.32) and taking into account (4.19) we arrive at

$$\begin{aligned} \frac{2\rho}{(\int_{\Omega} e^{W_1})^2} \int_{\Omega} e^{W_1} P w_2 dx \int_{\Omega} e^{W_1} \phi_{1n} dx &= -\frac{256}{\rho} \pi \log \delta_{2n} \langle \tilde{\phi}_{1n}, 1 \rangle_{L_2} \\ &\quad - \frac{64}{\rho} \pi \log \delta_{2n} \frac{\rho - 4\pi}{\int_{\Omega} e^z} \int_{\Omega} e^z \phi_{1n} dx + o(1) \end{aligned} \quad (4.67)$$

and, similarly to (4.58),

$$\int_{\Omega} \nabla h_{2n} \nabla P w_2 dx = o(1). \quad (4.68)$$

Passing to the limit in (4.60) and using (4.61), (4.64), (4.66), (4.67) and (4.68) we arrive at

$$\frac{-32\rho + 128\pi}{\rho} \log \delta_{2n} \langle \tilde{\phi}_{1n}, 1 \rangle_{L_2} + \frac{32}{\rho} \pi \log \delta_{2n} \frac{\rho - 4\pi}{\int_{\Omega} e^z} \int_{\Omega} e^z \phi_{1n} dx - 16\pi \gamma_2 = o(1)$$

by which, using Step 3, $\gamma_2 = 0$ and, consequently, by (4.59), $\gamma_1 = 0$.

Step 5. Conclusion.

We will show that a contradiction arises. According to Step 2 and Step 4 we have

$$\begin{aligned} \tilde{\phi}_{1n} &\rightarrow 0 \text{ weakly in } H_2(\mathbb{R}^2) \text{ and strongly in } L_2(\mathbb{R}^2), \\ \tilde{\phi}_{2n} &\rightarrow 0 \text{ weakly in } H_4(\mathbb{R}^2) \text{ and strongly in } L_4(\mathbb{R}^2), \end{aligned}$$

and

$$\phi_{1n} \rightarrow 0 \text{ weakly in } H_0^1(\Omega) \text{ and strongly in } L^q(\Omega) \text{ for any } q \geq 2.$$

By Lemma 3.1 we get

$$\begin{aligned} \frac{2\rho}{\int_{\Omega} e^{W_1}} \int_{\Omega} e^{W_1} \phi_{1n}^2 dx &= \int_{\Omega} e^{w_1} \phi_{1n}^2 dx + 2 \frac{\rho - 4\pi}{\int_{\Omega} e^z} \int_{\Omega} e^z \phi_{1n}^2 dx + o(1) \\ &= 8 \|\tilde{\phi}_{1n}\|_{L_2}^2 + 2 \frac{\rho - 4\pi}{\int_{\Omega} e^z} \int_{\Omega} e^z \phi_{1n}^2 dx + o(1) = o(1) \end{aligned} \quad (4.69)$$

and

$$2\lambda_n \int_{\Omega} e^{W_2} \phi_{2n}^2 dx = \int_{\Omega} |x|^2 e^{w_2} \phi_{2n}^2 dx + o(1) = 32 \|\tilde{\phi}_{2n}\|_{L_4}^2 + o(1) = o(1). \quad (4.70)$$

Moreover, recalling (4.19) and (4.32),

$$\frac{1}{\int_{\Omega} e^{W_1}} \int_{\Omega} e^{W_1} \phi_{1n} dx = o(1). \quad (4.71)$$

Next, we multiply the first and the second equations in (4.1) by ϕ_{1n} , we integrate over Ω and, using (4.69) and (4.71), we deduce

$$\begin{aligned} \int_{\Omega} |\nabla \phi_{1n}|^2 dx + \lambda_n \int_{\Omega} e^{W_2} \phi_{2n} \phi_{1n} dx &= o(1), \\ \int_{\Omega} \nabla \phi_{2n} \nabla \phi_{1n} dx - 2\lambda_n \int_{\Omega} e^{W_2} \phi_{2n} \phi_{1n} dx &= o(1), \end{aligned}$$

respectively. Combining the above identities we obtain

$$2 \int_{\Omega} |\nabla \phi_{1n}|^2 dx + \int_{\Omega} \nabla \phi_{2n} \nabla \phi_{1n} dx = o(1). \quad (4.72)$$

Similarly, we multiply the first and the second equations in (4.1) by ϕ_{2n} , we integrate over Ω and, using (4.70),

$$\begin{aligned} \int_{\Omega} \nabla \phi_{1n} \nabla \phi_{2n} dx - 2\rho \left[\frac{\int_{\Omega} e^{W_1} \phi_{1n} \phi_{2n} dx}{\int_{\Omega} e^{W_1} dx} - \frac{\int_{\Omega} e^{W_1} \phi_{2n} dx \int_{\Omega} e^{W_1} \phi_{1n} dx}{(\int_{\Omega} e^{W_1} dx)^2} \right] &= o(1), \\ \int_{\Omega} |\nabla \phi_{2n}|^2 dx + \rho \left[\frac{\int_{\Omega} e^{W_1} \phi_{1n} \phi_{2n} dx}{\int_{\Omega} e^{W_1} dx} - \frac{\int_{\Omega} e^{W_1} \phi_{2n} dx \int_{\Omega} e^{W_1} \phi_{1n} dx}{(\int_{\Omega} e^{W_1} dx)^2} \right] &= o(1), \end{aligned}$$

by which

$$2 \int_{\Omega} |\nabla \phi_{2n}|^2 dx + \int_{\Omega} \nabla \phi_{2n} \nabla \phi_{1n} dx = o(1). \quad (4.73)$$

Summing up (4.72) and (4.73),

$$\int_{\Omega} |\nabla \phi_{1n}|^2 dx + \int_{\Omega} |\nabla \phi_{2n}|^2 dx \leq 2 \int_{\Omega} |\nabla \phi_{1n}|^2 dx + 2 \int_{\Omega} |\nabla \phi_{2n}|^2 dx + 2 \int_{\Omega} \nabla \phi_{2n} \nabla \phi_{1n} dx = o(1).$$

A contradiction arises with (4.2). \square

5. THE CONTRACTION ARGUMENT: PROOF OF THEOREM 1.1 AND THEOREM 2.2

Once we have studied the solvability of the linearized problem, we are in position to prove Theorem 2.2.

First let us rewrite problem (2.3) in a more convenient way. For any $p > 1$, let

$$i_p^* : L^p(\Omega) \rightarrow H_0^1(\Omega)$$

be the adjoint operator of the embedding $i_p : H_0^1(\Omega) \hookrightarrow L^{\frac{p}{p-1}}(\Omega)$, i.e. $u = i_p^*(v)$ if and only if $-\Delta u = v$ in Ω , $u = 0$ on $\partial\Omega$. We point out that i_p^* is a continuous mapping, namely

$$\|i_p^*(v)\| \leq c_p \|v\|_p, \text{ for any } v \in L^p(\Omega), \quad (5.1)$$

for some constant c_p which depends on Ω and p . Then, setting $u := (u_1, u_2)$ and $i_p^*(u) := (i_p^*(u_1), i_p^*(u_2))$, problem (2.3) is equivalent to

$$u = i_p^*(F(u)) \quad (5.2)$$

where

$$F(u) := (2\rho g(u_1) - \lambda f(u_2), 2\lambda f(u_2) - \rho g(u_1))$$

and

$$f(u_2) := e^{u_2} \quad \text{and} \quad g(u_1) := \frac{e^{u_1}}{\int_{\Omega} e^{u_1} dx}.$$

Next we denote by $L : H_0^1(\Omega) \times H_0^1(\Omega) \rightarrow H_0^1(\Omega) \times H_0^1(\Omega)$ the linear operator defined by

$$L(\phi) := i_p^* (F'(W_\lambda)\phi) - \phi, \quad \phi = (\phi_1, \phi_2),$$

where

$$F'(W_\lambda)(\phi) = \begin{pmatrix} 2\rho \left[\frac{e^{W_{1\lambda}} \phi_1}{\int_{\Omega} e^{W_{1\lambda}}} - \frac{e^{W_{1\lambda}} \int_{\Omega} e^{W_{1\lambda}} \phi_1 dx}{(\int_{\Omega} e^{W_{1\lambda}})^2} \right] - \lambda e^{W_{2\lambda}} \phi_2 \\ -\rho \left[\frac{e^{W_{1\lambda}} \phi_1}{\int_{\Omega} e^{W_{1\lambda}}} - \frac{e^{W_{1\lambda}} \int_{\Omega} e^{W_{1\lambda}} \phi_1 dx}{(\int_{\Omega} e^{W_{1\lambda}})^2} \right] + 2\lambda e^{W_{2\lambda}} \phi_2 \end{pmatrix}.$$

Notice that problem (4.1) reduces to

$$L[\phi] = h, \quad \phi, h \in \mathcal{H}_k \times \mathcal{H}_k. \quad (5.3)$$

As a consequence of Proposition 4.1 we derive the invertibility of L .

Proposition 5.1. *For any $p \in (1, 2)$ there exist $\lambda_0 > 0$ and $C > 0$ such that for any $\lambda \in (0, \lambda_0)$ and for any $h = (h_1, h_2) \in \mathcal{H}_k \times \mathcal{H}_k$ there is a unique solution $\phi = (\phi_1, \phi_2)$ to the problem (5.3). In particular, L is invertible; moreover,*

$$\|L^{-1}\| \leq C |\log \lambda|.$$

Proof. Observe that the operator $\phi \mapsto i_p^*(F'(W_\lambda)\phi)$ is a compact operator in $\mathcal{H}_k \times \mathcal{H}_k$. Let us consider the case $h = 0$, and take $\phi \in \mathcal{H}_k \times \mathcal{H}_k$ with $L[\phi] = 0$. In other words, ϕ solves the system (4.1) with $h_1 = h_2 = 0$. Proposition 4.1 implies $\phi \equiv 0$. Then, Fredholm's alternative implies the existence and uniqueness result.

Once we have existence, the norm estimate follows directly from Proposition 4.1. \square

The nonlinear problem. Recall that we are interested in finding a solution u of (5.2) with $u = W_\lambda + \phi$, for some small $\phi \in \mathcal{H}_k \times \mathcal{H}_k$. In what follows we denote by $N : H_0^1(\Omega) \times H_0^1(\Omega) \rightarrow H_0^1(\Omega) \times H_0^1(\Omega)$ the nonlinear operator

$$N(\phi) := i_p^* (F(W_\lambda + \phi) - F(W_\lambda) - F'(W_\lambda)\phi).$$

Therefore, problem (5.2) turns out to be equivalent to the problem

$$N(\phi) + L(\phi) = \tilde{R}_\lambda, \quad \phi \in \mathcal{H}_k \times \mathcal{H}_k \quad (5.4)$$

with

$$\tilde{R}_\lambda = W_\lambda - i_p^*(F(W_\lambda)).$$

Observe that $\tilde{R}_\lambda = i_p^*(R_\lambda)$, where R_λ is given in (3.6).

The following lemma will be of use in the following:

Lemma 5.2. *For any $p \geq 1$, $r_0 > 0$ and $\eta > 0$ there exist $\lambda_0 > 0$ and $C > 0$ such that, for any $\lambda \in (0, \lambda_0)$ and for any $u \in H_0^1(\Omega)$ with $\|u\| \leq r_0$,*

- a) $\|\lambda f'(W_2 + u) v\|_p \leq C \lambda^{\frac{1-p}{p}-\eta} \|v\| \quad \forall v \in H_0^1(\Omega),$
- b) $\|g'(W_1 + u) v\|_p \leq C \lambda^{2\frac{1-p}{p}-\eta} \|v\| \quad \forall v \in H_0^1(\Omega),$
- c) $\|\lambda f''(W_2 + u) v z\|_p \leq C \lambda^{\frac{1-p}{p}-\eta} \|v\| \|z\| \quad \forall v, z \in H_0^1(\Omega),$
- d) $\|g''(W_1 + u) v z\|_p \leq C \lambda^{3\frac{1-p}{p}-\eta} \|v\| \|z\| \quad \forall v, z \in H_0^1(\Omega).$

Proof. The proof of this lemma is basically contained in [8, Lemma 4.7]; however, we reproduce it here for the sake of completeness. To start with, easy computations lead to the estimate:

$$\|e^{w_1}\|_p = O(\lambda^{\frac{1-p}{p}}), \quad \| |x|^2 e^{w_2} \|_p = O(\lambda^{\frac{1-p}{p}}) \quad \forall p \geq 1. \quad (5.5)$$

By Lemma 3.1 we conclude that:

$$\|e^{W_1}\|_p = O(\lambda^{\frac{1-p}{p}}), \quad \|\lambda e^{W_2}\|_p = O(\lambda^{\frac{1-p}{p}}) \quad \forall p \geq 1. \quad (5.6)$$

We give the complete proof for inequalities c), d), the others being easier. We point out that by Hölder's inequality with $\frac{1}{q} + \frac{1}{r} + \frac{1}{s} + \frac{1}{t} = 1$,

$$\begin{aligned} \|\lambda f''(W_2 + u) v z\|_p &\leq \|\lambda e^{W_2}\|_{pq} \|e^u\|_{pr} \|v\|_{ps} \|z\|_{pt} \\ &\quad (\text{we use the continuity of } H_0^1(\Omega) \hookrightarrow L^p(\Omega)) \\ &\leq C \|\lambda e^{W_2}\|_{pq} \|e^u\|_{pr} \|v\| \|z\| \\ &\quad (\text{we use Lemma 2.3}) \\ &\leq C \|\lambda e^{W_2}\|_{pq} e^{\frac{pr}{16\pi} \|u\|^2} \|v\| \|z\| \\ &\leq C \lambda^{\frac{1-pq}{pq}} e^{\frac{pr}{16\pi} \|u\|^2} \|v\| \|z\|. \end{aligned}$$

It suffices now to choose $q > 1$ sufficiently small to obtain c). Moreover

$$\begin{aligned} g''(W_1 + u)[v, z] &= \frac{e^{W_1+u}}{\int_{\Omega} e^{W_1+u}} v z - \frac{e^{W_1+u}}{\left(\int_{\Omega} e^{W_1+u}\right)^2} v \int_{\Omega} e^{W_1+u} z - \frac{e^{W_1+u}}{\left(\int_{\Omega} e^{W_1+u}\right)^2} z \int_{\Omega} e^{W_1+u} v \\ &\quad - \frac{e^{W_1+u}}{\left(\int_{\Omega} e^{W_1+u}\right)^2} \int_{\Omega} e^{W_1+u} v z + 2 \frac{e^{W_1+u}}{\left(\int_{\Omega} e^{W_1+u}\right)^3} \int_{\Omega} e^{W_1+u} v \int_{\Omega} e^{W_1+u} z. \end{aligned}$$

We use Hölder's inequalities with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, $\frac{1}{a} + \frac{1}{b} + \frac{1}{d} = 1$, and $\frac{1}{q} + \frac{1}{r} + \frac{1}{s} + \frac{1}{t} = 1$ and α, a, q sufficiently close to 1: we obtain

$$\begin{aligned} \|g''(W_1 + u)[v, z]\|_p &\leq \frac{\|e^{W_1}\|_{pq} \|e^u\|_{pr} \|v\|_{ps} \|z\|_{pt}}{\|e^{W_1+u}\|_1} + 2 \frac{\|e^{W_1}\|_{pa}^2 \|e^u\|_{pb}^2 \|v\|_{pd} \|z\|_{pd}}{\|e^{W_1+u}\|_1^2} \\ &\quad + \frac{\|e^{W_1}\|_{p\alpha} \|e^u\|_{p\beta} \|e^{W_1}\|_{pq} \|e^u\|_{pr} \|v\|_{ps} \|z\|_{pt}}{\|e^{W_1+u}\|_1^2} \\ &\quad + 2 \frac{\|e^{W_1}\|_{p\alpha} \|e^u\|_{p\beta} \|e^{W_1}\|_{pa}^2 \|e^u\|_{pb}^2 \|v\|_{pd} \|z\|_{pd}}{\|e^{W_1+u}\|_1^3} \\ &\quad \text{(we use the continuity of } H_0^1(\Omega) \hookrightarrow L^p(\Omega) \text{ and Lemma 2.3)} \\ &\leq C\lambda^{3\frac{1-p}{p}-\eta} e^{c\|u\|^2} \|v\| \|z\|. \end{aligned}$$

It is important to point out that

$$\|e^{W_1+u}\|_1 \geq c. \quad (5.7)$$

Indeed, by Lemma 3.1, it suffices to show that:

$$\|(e^{w_1} + e^z)e^u\|_1 \geq c. \quad (5.8)$$

Clearly, $\int_{\Omega} e^{w_1} e^u \geq 0$. Moreover,

$$\int_{\Omega} e^z e^u dx \geq c \int_{\Omega} e^u dx \geq c \int_{\Omega} e^{-|u|} dx \geq c|\Omega| e^{-\frac{1}{|\Omega|} \int_{\Omega} |u|} \geq c|\Omega| e^{-C}.$$

In the above estimates the Jensen's inequality has been used. \square

Lemma 5.3. *For any $\eta > 0$, $r_0 > 0$ there exist $\lambda_0 > 0$ and $C > 0$ such that for any $\lambda \in (0, \lambda_0)$ and for any $\phi, \psi \in H_0^1(\Omega) \times H_0^1(\Omega)$ with $\|\phi\|, \|\psi\| \leq r_0$*

$$\|N(\phi)\| \leq C\lambda^{-2\eta} \|\phi\|^2 \quad (5.9)$$

and

$$\|N(\phi) - N(\psi)\| \leq C\lambda^{-2\eta} \|\phi - \psi\| (\|\phi\| + \|\psi\|). \quad (5.10)$$

Proof. Let us remark that (5.9) follows by choosing $\psi = 0$ in (5.10). Let us prove (5.10). First of all, we point out that for any $p > 1$

$$\|N(\phi) - N(\psi)\| \leq c_p \|F(W_{\lambda} + \phi) - F(W_{\lambda} + \psi) - F'(W_{\lambda})(\phi - \psi)\|_p.$$

We apply the mean value theorem ([1, Theorem 1.8]) to the map: $\varphi \mapsto F(\varphi + W_{\lambda}) - F'(W_{\lambda})\varphi \in L^p(\Omega) \times L^p(\Omega)$, with $\varphi \in H_0^1(\Omega) \times H_0^1(\Omega)$. Then, there exists $\theta \in (0, 1)$ such that

$$\|F(W_{\lambda} + \phi) - F(W_{\lambda} + \psi) - F'(W_{\lambda})(\phi - \psi)\|_p \leq \| [F'(W_{\lambda} + \theta\phi + (1-\theta)\psi) - F'(W_{\lambda})](\phi - \psi) \|_p.$$

We apply again the mean value theorem to the map $\varphi \mapsto F'(\varphi + W_\lambda)(\phi - \psi)$; there exists $\sigma \in (0, 1)$ such that

$$\begin{aligned} & \| [F'(W_\lambda + \theta\phi + (1 - \theta)\psi) - F'(W_\lambda)] (\phi - \psi) \|_p \\ & \leq \| F''(W_\lambda + \sigma(\theta\phi + (1 - \theta)\psi))(\theta\phi + (1 - \theta)\psi)(\phi - \psi) \|_p. \end{aligned}$$

Taking into account that

$$F''(u) = (2\rho g''(u_1) - \lambda f''(u_2), 2\lambda f''(u_2) - \rho g''(u_1)),$$

Lemma 5.2 allows us to conclude by choosing p sufficiently close to 1. □

Now we are able to solve problem (5.4).

Proposition 5.4. *For any $\varepsilon \in (0, \frac{1}{4})$ there exists $\lambda_0 > 0$ such that for any $\lambda \in (0, \lambda_0)$ there is a unique $\phi_\lambda \in \mathcal{H}_k \times \mathcal{H}_k$ satisfying (5.4) and*

$$\|\phi_\lambda\| \leq \lambda^{\frac{1}{4}-\varepsilon}.$$

Proof. Problem (5.4) can be solved via a contraction mapping argument. Indeed, in virtue of Proposition 5.1, we can introduce the map

$$T(\phi) := L^{-1}(\tilde{R}_\lambda - N(\phi)), \quad \phi \in \mathcal{H}_k \times \mathcal{H}_k.$$

By (5.1) and Lemma 3.2, recalling that $\tilde{R}_\lambda = i_p^*(R_\lambda)$, we have:

$$\|\tilde{R}_\lambda\| = O(\lambda^{\frac{1}{4} \frac{2-p}{p}}) \quad \forall p > 1,$$

or, equivalently,

$$\|\tilde{R}_\lambda\| = O(\lambda^{\frac{1}{4}-\eta}) \quad \forall \eta > 0. \tag{5.11}$$

We claim that T is a contraction map over the ball

$$\left\{ \phi \in \mathcal{H}_k \times \mathcal{H}_k : \|\phi\| \leq \lambda^{\frac{1}{4}-\varepsilon} \right\} \tag{5.12}$$

provided λ is small enough. Indeed, using (5.11) and Lemma 5.3, fixed $0 < \eta < \min\{\varepsilon, \frac{1}{8} - \frac{\varepsilon}{2}\}$, we have

$$\|T(\phi)\| \leq C |\log \lambda| (\lambda^{\frac{1}{4}-\eta} + \lambda^{-2\eta} \|\phi\|^2) < \lambda^{\frac{1}{4}-\varepsilon}$$

and

$$\|T(\phi) - T(\psi)\| \leq C |\log \lambda| \lambda^{-2\eta} (\|\phi\| + \|\psi\|) \|\phi - \psi\| < \frac{1}{2} \|\phi - \psi\|.$$

□

Proof of Theorem 2.2. It follows immediately from Proposition 5.4, since, as we have already observed, problem (2.3) is equivalent to (5.4). □

Proof of Theorem 1.1. By [3, 25] (see remark 2.1), assumption (H) is satisfied, and hence Theorem 2.2 provides us with a solution $u_\lambda = W_\lambda + \phi_\lambda$ of (1.1) with

$$\rho_1 = \rho, \quad \rho_2 = \rho_{2\lambda} = \lambda \int_{\Omega} e^{u_{2\lambda}} dx.$$

Clearly, by (1.3) and (2.5),

$$v_{1\lambda} = \frac{2u_{1\lambda} + u_{2\lambda}}{3} = \frac{1}{2}(Pw_1 + z) + o(1), \quad v_{2\lambda} = \frac{2u_{2\lambda} + u_{1\lambda}}{3} = \frac{1}{2}Pw_2 + o(1)$$

in the H^1 -sense and the expansions of Theorem 1.1 follow from (2.8), recalling also (2.7). Moreover, using Hölder's inequality with $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$ and (5.6),

$$\begin{aligned} \|e^{u_{1\lambda}} - e^{W_{1\lambda}}\|_1 &= \|e^{W_{1\lambda} + \phi_{1\lambda}} - e^{W_{1\lambda}}\|_1 = \int_{\Omega} e^{W_{1\lambda}} |e^{\phi_{1\lambda}} - 1| dx \leq \int_{\Omega} e^{W_{1\lambda}} e^{|\phi_{1\lambda}|} |\phi_{1\lambda}| dx \\ &\leq C \|e^{W_{1\lambda}}\|_a \|e^{\phi_{1\lambda}}\|_b \|\phi_{1\lambda}\|_c = o(1), \end{aligned}$$

if a is chosen sufficiently close to 1. Similarly,

$$\|\lambda e^{u_{2\lambda}} - \lambda e^{W_{2\lambda}}\|_1 = \|\lambda e^{W_{2\lambda} + \phi_{2\lambda}} - \lambda e^{W_{2\lambda}}\|_1 = o(1).$$

Then, by Lemma 3.1, for every $r > 0$

$$\begin{aligned} \frac{\rho_1}{\int_{\Omega} e^{u_{1\lambda}}} \int_{B(0,r)} e^{u_{1\lambda}} dx &= \frac{\rho}{\int_{\Omega} e^{W_{1\lambda}}} \int_{B(0,r)} e^{W_{1\lambda}} dx + o(1) \\ &= \frac{1}{2} \int_{B(0,r)} e^{w_1} dx + \frac{\rho - 4\pi}{\int_{\Omega} e^z} \int_{B(0,r)} e^z dx + o(1) \\ &\rightarrow 4\pi + \frac{\rho - 4\pi}{\int_{\Omega} e^z} \int_{B(0,r)} e^z dx \end{aligned}$$

as $\lambda \rightarrow 0$. If we now make $r \rightarrow 0$, we obtain that $\sigma_1 = 4\pi$. Analogously, if we use Lemma 3.1,

$$\lambda \int_{B(0,r)} e^{u_{2\lambda}} dx = \lambda \int_{B(0,r)} e^{W_{2\lambda}} dx + o(1) = \frac{1}{2} \int_{B(0,r)} |x|^2 e^{w_2} dx + o(1) \rightarrow 8\pi,$$

as $\lambda \rightarrow 0$, by which $\sigma_2 = 8\pi$. Moreover, in this case there is no global mass since, again by Lemma 3.1,

$$\rho_{2\lambda} = \lambda \int_{\Omega} e^{u_{2\lambda}} dx = \frac{1}{2} \int_{\Omega} |x|^2 e^{w_2} dx + o(1) \rightarrow 8\pi.$$

This concludes the proof. \square

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